DEFECTS IN SEMILINEAR WAVE EQUATIONS AND TIMELIKE MINIMAL SURFACES IN MINKOWSKI SPACE

ROBERT L. JERRARD

ABSTRACT. We study semilinear wave equations with Ginzburg-Landau type nonlinearities multiplied by a factor ε^{-2}, where $\varepsilon > 0$ is a small parameter. We prove that for suitable initial data, solutions exhibit energy concentration sets that evolve approximately via the equation for timelike Minkowski minimal surfaces, as long as the minimal surface remains smooth. This gives a proof of predictions made, on the basis of formal asymptotics and other heuristic arguments, by cosmologists studying cosmic strings and domain walls, as well as by applied mathematicans.

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1. INTRODUCTION

In this paper we prove that if Γ is a timelike minimal surface in Minkowski space $\mathbb{R}^{1+N}$ of codimension $k = 1$ or 2, smooth in a time interval $(-T, T)$, then for suitable initial data, solutions $u : \mathbb{R}^{1+N} \to \mathbb{R}^k$, $N > k$ of the equation

$$\Box u + \frac{1}{\varepsilon^2} f(u) = 0, \qquad 0 < \varepsilon \ll 1 \tag{1.1}$$

exhibit an energy concentration set that approximately follows Γ, at least up to time T. Here the model nonlinearity is $f(u) = (|u|^2 - 1)u$ in low dimensions; in higher dimensions, we take f to be a qualitatively similar nonlinearity satisfying growth conditions that leave the equation (1.1) globally well-posed; see (1.9), (1.19) for precise assumptions.

Our main motivation for this work comes from the very rich mathematical literature on corresponding questions about elliptic and parabolic analogs of (1.1), which have been studied in great detail for about the past 30 years. In the elliptic case, these past results establish deep connections between energy concentration sets in solutions $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^k$ of the equation

$$- \Delta u + \frac{1}{\varepsilon^2} f(u) = 0, \qquad 0 < \varepsilon \ll 1 \tag{1.2}$$

and (Euclidean) minimal surfaces of codimension k in Ω. Similarly, the parabolic equation

$$u_t - \Delta u + \frac{1}{\varepsilon^2} f(u) = 0, \qquad 0 < \varepsilon \ll 1, \qquad u : (0, T) \times \mathbb{R}^N \to \mathbb{R}^k \tag{1.3}$$

is related to the geometric evolution problem of codimension k motion by mean curvature. Our results address the natural question of whether any parallel results hold, relating the

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semilinear for wave equation (1.1) to the timelike Minkowski minimal surface problem, which is a geometric wave equation.

It turns out that this question is also relevant to the description of cosmological domain walls ($k = 1$) and strings ($k = 2$) ; see Kibble [24] for a seminal early paper and Vilenkin and Shellard [41] for an in-depth survey of a large body of work on related questions. The questions we study have also been addressed in the applied math literature by Neu [32], with some generalizations considered by Nepomnyashchy and Rotstein [34]. We will not say any more about any of these applications in this paper, except to note that our main results can be described as giving a rigorous derivation, in the relatively simple and physically unrealistic setting of a scalar particle described by (1.1), of the laws of motion for cosmic strings and domain walls, deduced formally by cosmologists over 30 years ago.

1.1. **mathematical background.** We first review results about the elliptic and parabolic equations (1.2) and (1.3). Throughout this discussion we consider the model nonlinearity $f(u) = (|u|^2 - 1)u$.

In the elliptic case, and when $k = 1$ (so that (1.2) is a scalar equation), the general heuristic principle underlying essentially every work we know of is that

$$u \approx q(\frac{d}{\varepsilon}) \tag{1.4}$$

where $q : \mathbb{R} \to \mathbb{R}$ solves

$$-q'' + f(q) = 0, \qquad q(0) = 0, \quad q(x) \to \pm 1 \text{ as } x \to \pm\infty \tag{1.5}$$

and $d : \Omega \to \mathbb{R}$ is the signed distance function to a *minimal* hypersurface $\Gamma \subset \Omega$, so that d is characterized near Γ by the properties

$$d = 0 \text{ on } \Gamma, \qquad |\nabla d|^2 = 1 \text{ near } \Gamma \tag{1.6}$$

and Γ satisfies

$$\text{(Euclidean) mean curvature } = 0. \tag{1.7}$$

There are a vast number of results establishing various forms of these assertions. Roughly speaking, these fall into two families. The first (see for example Modica [31] or Hutchinson and Tonegawa [16]) employ variational and measure theoretic methods, together with elliptic estimates, to characterize the limiting behavior of sequences of solutions as $\varepsilon \to 0$. These proofs generally establish some form of what is called equipartition of energy, which can be viewed as a weak form of the description (1.4). The second family of proofs (see for example Pacard and Ritoré [33]) employ Liapunov-Schmidt reduction and related arguments, relying ultimately on the the implicit function theorem and control of the spectrum of some linearized operator. These arguments yield existence results that give very precise descriptions, in the spirit of (1.4), of the solutions that are constructed.

In the $k = 1$ scalar case of the parabolic equation (1.3), more or less the same heuristic (1.4), (1.5) holds, except that now d is a function of t and x, and for every t, $d(t, \cdot)$ is the signed distance function from a hypersurface Γ_t, so that

$$d(t, \cdot) = 0 \text{ on } \Gamma_t \text{ and } |\nabla_x d(t,)|^2 = 1 \text{ near } \Gamma_t,$$

with $\Gamma := \cup_{t>0}\{t\} \times \Gamma_t \subset (0,T) \times \mathbb{R}^N$ satisfying

$$\text{velocity} = \text{ mean curvature.} \tag{1.8}$$

Different versions of this result have been established by a variety of proofs, including linearization techniques (see de Mottoni and Schatzmann [13]) which establish a strong form of (1.4), but are valid only locally in t; maximum principle arguments which ultimately rely on an ansatz based on (1.4) to build sub- and super-solutions (see [11, 14]), or which employ a change of variables motivated by (1.4) and techniques for weak passage to limits [7]; and measure theoretic methods combined with parabolic estimates as in Ilmanen [17], in which (1.4) appears in the weak form of assertions about equipartition of energy. The maximum principle and measure theoretic arguments give weaker descriptions that are however valid globally in t, with (1.8) understood in a weak sense.

In vector-valued $k = 2$ case, for both the elliptic (1.2) and parabolic (1.3) systems, we do not know of any characterization as precise as (1.4); obstacles to such results include the difficulty of describing rotational degrees of freedom, and the related poor behavior of the spectrum of certain linearized operators. But there are a number of results showing in various degrees of generality for solutions of (1.2) (including among others [27, 6, 2]) and (1.3) (see [3, 28, 7] for example) with suitable energy bounds, that energy concentrates around a codimension 2 submanifold Γ satisfying (1.7), respectively (1.8). These results generally employ elliptic or parabolic estimates, some of which are extremely delicate, in combination with measure theoretic arguments, and they provide information, customarily phrased in the language of varifold convergence, about the precise way in which energy concentrates around the codimension 2 surface Γ.

All results about (1.2) and (1.3) rely very heavily on tools that are not available for hyperbolic equations, such as maximum principles (in the scalar case) and elliptic or parabolic regularity. Thus they do not give much indication of how to proceed for the nonlinear wave equation (1.1). We know of only two partial exceptions to this rule. First, there is no abstract reason that linearization arguments should be impossible in the hyperbolic setting; they appear however to be hard to carry through. Second, a number of papers, starting with [10], study (1.3) using weighted energy estimates. In particular, we mention an argument presented by Soner in a 1995 lecture series [38] for the scalar parabolic equation (1.3), and developed in [20, 25] for parabolic systems. This argument relies on a rather straightforward but remarkable computation of $\frac{d}{dt}\int_{\mathbb{R}^N} \zeta e_\varepsilon(u)dx$, where $e_\varepsilon(u)$ is a natural energy density associated with a solution u of (1.3), and ζ is a smooth function such that $\zeta(t,x) = \frac{1}{2}\operatorname{dist}(x,\Gamma_t)^2$ near Γ_t, where the latter solves (1.8). This calculation certainly uses the parabolic character of (1.3), but it is not clear if it uses it in a really essential way. Indeed, our main proofs originated as an attempt to develop an analog of this argument in the hyperbolic setting.

Much less work has been done on the hyperbolic equation (1.1) than on its elliptic and parabolic counterparts. The few papers that we are aware of mostly study situations rather different from those we consider here, including:

- works [19, 26] that characterize the behavior of solutions of (1.1) in the limit $\varepsilon \to 0$ in the case $N = k = 2$, for the model nonlinearity $f(u) = (|u|^2 - 1)u$.

- a paper of Gustafson and Sigal [15] that studies the Maxwell-Higgs model, in which (1.1), with the model nonlinearity $f(u) = (|u|^2 - 1)u$, is coupled to a electromagnetic field, when $N = k = 2$ and $0 < \varepsilon \ll 1$.
- work of Stuart [39] studying an equation of the form (1.1) on a Lorentzian manifold and with a focussing nonlinearity, for $0 < \varepsilon \ll 1$, see also [40].

In all these papers, energy concentrates around points, know as vortices or quasiparticles depending on the situation, and these points evolve according to an ODE. These results are valid only as long as the points remain separated from each other. The fact that points are geometrically very simple objects makes the analysis easier in some ways than in the problems we consider here, where the same role is now played by submanifolds of dimension $n \geq 1$. An additional significant simplifying factor in all the papers cited above, except those of Stuart, is that they study a scaling in which vortices move at subrelativistic velocites, that is, velocities that tend to 0 as $\varepsilon \to 0$.

It is also worth mentioning work [12] of Cuccagna that studies (1.1) in $\mathbb{R}^{1+3}$ with $\varepsilon = 1$ and establishes scattering for initial data $(u, u_t)|_{t=0}$ a small, very smooth perturbation of $(q(x^3), 0)$. This can be seen as an analog for (1.1) of results [29, 9] that establish scattering for solutions of the timelike Minkowski minimal surface problem with initial data that is a small, perturbation of a motionless hyperplane.

As far as we know, the only work of rigorous mathematics that addresses exactly the questions we consider here is a recent preprint of Bellettini, Novaga, and Orlandi [5]. Its main result identifies some conditions that, if they could be verified, would suffice to imply that a varifold obtained from a sequence of solutions (u_ε) of (1.1) satisfying natural energy bounds is stationary with respect to the Minkowski inner product structure. These conditions include lower density bounds as well as, roughly speaking, some quite strong constraints on the limiting tangent space. The results we obtain here are stronger than those projected in [5], as discussed in Remark 1.6

1.2. **new results.** In many ways our results follow the pattern described above. In the case $k = 1$ of a scalar equation, as in earlier work on the elliptic and parabolic problems, we obtain, for suitable initial data, a description of solutions of (1.1) parallel to (1.4), (1.5), (1.6), (1.7), with the Euclidean metric replaced by the Minkowski metric in the last two identities. And in the case $k = 2$, we prove that for solutions of (1.1) with suitable initial data, energy concentrates around a codimension 2 surface Γ that satisfies (1.7), again the Euclidean metric replaced by the Minkowski metric. We also give a precise description of the way that this concentration occurs; in fact we obtain this description in the case $k = 1$ as well.

The strongest results (for example Bethuel, Orlandi and Smets [7]) on the parabolic equation (1.3) hold globally for $t > 0$, and assume only natural energy bounds on the initial data. Our results, by contrast, are valid only locally in t — that is, as long as the surface Γ remains smooth — and require rather special initial data. We note however that results like those we obtain are almost certainly *not true* globally in t or for general initial data.

In all our results, we take the timelike minimal surface Γ to have the topology of $(-T, T) \times \mathbb{T}^n$, where $n = N - k$. This covers the important example of a closed string in $\mathbb{R}^3$, when $k = 2$. In fact, we view the global topology of Γ as relatively unimportant,

since our results are in some sense local, and since both the semilinear wave equation (1.1) and the timelike minimal surface equation enjoy finite propagation speed, In any case, our methods should extend to $\Gamma \cong (-T,T)\times M$ for more general M.

Quite general results in Milbredt [30] imply in particular the local existence of smooth timelike minimal surfaces Γ given smooth data at $t=0$.

In the scalar case, we assume that the nonlinearity f in (1.1) has the form $f = F'$, where $F:\mathbb{R}\to\mathbb{R}$ is a smooth function such that

$$F(\pm 1) = 0, \qquad c(1-|s|)^2 \le F(s) \tag{1.9}$$

We also assume that f grows sufficiently slowly that (1.1) is globally well-posed in $\dot H^1\times L^2$. We may take $f(u) = (u^2-1)u$ if $N\le 4$.

In the statement of our results we use the notation

$$e_\varepsilon(u) := \frac{1}{2}(u_t^2 + |\nabla u|^2) + \frac{1}{\varepsilon^2}F(u) \tag{1.10}$$

and

$$\kappa_1 \ := \ \int_{-1}^{1}\sqrt{2F(s)}\ ds. \tag{1.11}$$

One can think of κ_1 as a constant related to the surface tension of an interface. Our main results in the scalar case can be summarized as follows:

Theorem 1. *Let $\Gamma\subset(-T,T)\times\mathbb{R}^N$ be a smooth timelike minimal hypersurface. Let $\Gamma\cap(\{t\}\times\mathbb{R}^N) := \Gamma_t$, and assume that for every $t\in(-T,T)$, Γ_t is diffeomorphic to the torus $\mathbb{T}^n$, $n = N-1$.*

Then given $T_0 < T$, there exists a neighborhood $\mathcal{N}$ of Γ in $(-T_0,T_0)\times\mathbb{R}^N$ in which there exists a smooth solution $d:\mathcal{N}\to\mathbb{R}$ of the problem

$$d = 0 \ \text{on}\ \Gamma, \qquad -d_t^2 + |\nabla d|^2 = 1 \ \text{near}\ \Gamma. \tag{1.12}$$

(In other words, d is the signed Minkowski *distance to Γ, compare* (1.6)*.) Moreover, there exists a solution u of* (1.1) *(with f as described above) such that for any $T_0 < T$,*

$$\|u - q(\frac{d}{\varepsilon})\|_{L^2(\mathcal{N})} \le C\sqrt{\varepsilon}, \tag{1.13}$$

where q solves (1.5)*, and*

$$\int_{\mathcal{N}} d^2\ e_\varepsilon(u)\ dt\ dx \ + \ \int_{[(-T_0,T_0)\times\mathbb{R}^N]\setminus\mathcal{N}} e_\varepsilon(u)\ dt\ dx \le C\varepsilon \tag{1.14}$$

In addition, if $\mathcal{T}_\varepsilon(u) = (\mathcal{T}^\alpha_{\varepsilon,\beta}(u))^N_{\alpha,\beta=0}$ and $\mathcal{T}(\Gamma) = (\mathcal{T}^\alpha_\beta(\Gamma))^N_{\alpha,\beta=0}$ denote the energy-momentum tensors for u and Γ (defined in (2.8) *and* (2.9) *respectively) then*

$$\left\| \frac{\varepsilon}{\kappa_1}\mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\right\|_{W^{-1,1}((-T_0,T_0)\times\mathbb{R}^N)} \le C\varepsilon \tag{1.15}$$

In all these conclusions, $C = C(T_0,\Gamma)$ is independent of ε.

Remark 1.1. The definitions imply that $\mathcal{T}^0_{\varepsilon,0}(u) = e_\varepsilon(u)$, and that $\mathcal{T}^0_0(\Gamma)$ is a measure supported on Γ and defined by

$$\int f(t,x) d\mathcal{T}^0_0 = \int_{-T}^{T} \int_{\Gamma_t} f(t,x)(1-V^2)^{-1/2} \mathcal{H}^n(dx)\ dt$$

where $V(t,x)$ denotes the (euclidean) normal velocity of Γ at a point $(t,x) \in \Gamma$. We can denote this measure by $(1-V^2)^{-1/2}(\mathcal{H}^n \llcorner \Gamma_t) \otimes dt$. The conclusion (1.15) thus implies in particular that

$$\left\| \frac{\varepsilon}{\kappa_1} e_\varepsilon(u) \ - \ (1-V^2)^{-1/2}(\mathcal{H}^n \llcorner \Gamma_t) \otimes dt \right\|_{W^{-1,1}((-T_0,T_0)\times\mathbb{R}^N)} \le C\varepsilon. \tag{1.16}$$

A parallel remark holds for conclusion (1.21) of Theorem 2 below.

Remark 1.2. Our assumptions are satisfied for example if

$$u(0,x) = q(\frac{d(0,x)}{\varepsilon}), \qquad u_t(0,x) = \frac{1}{\varepsilon} q'(\frac{d(0,x)}{\varepsilon}) d_t(0,x) \tag{1.17}$$

in a neighborhood $\mathcal{N}_0$ of Γ_0, and if

$$\int_{\{0\}\times(\mathbb{R}^N \setminus \mathcal{N}_0)} e_\varepsilon(u)\ dx \le \varepsilon. \tag{1.18}$$

See Lemma 4 for details.

In the vector case, we can again take $f(u) = (|u|^2-1)u$ if $N \le 4$, or in other words, $f = \nabla_u F$, for $F(u) = \frac{1}{4}(|u|^2-1)^2$. More generally, we require of f only that the equation (1.1) be globally well-posed in $\dot H^1 \times L^2$, and that $f = \nabla_u F$, where

$$c(1-|u|)^2 \le F(u) \le C(1-|u|)^2 \quad \text{for } |u| \le 2 \qquad \text{and } F(u) \ge c > 0 \text{ for } |u| \ge 2. \tag{1.19}$$

We summarize our results in the vector $k=2$ case in the following:

Theorem 2. *Let $\Gamma \subset (-T,T)\times\mathbb{R}^N$ be a smooth timelike minimal surface of codimension $k=2$. Let $\Gamma \cap (\{t\}\times\mathbb{R}^N) := \Gamma_t$, and assume that for every $t \in (-T,T)$, Γ_t is diffeomorphic to the torus $\mathbb{T}^n$, $n = N-2 \ge 1$.*

Then there exists a solution for (1.1) (with $k=2$) such that for any $T_0 < T$, there is a constant C such that

$$\int_{(-T_0,T_0)\times\mathbb{R}^N} \tilde d^2 e_\varepsilon(u)\ dt\ dx \le C \tag{1.20}$$

where $\tilde d(t,x) = \min\{1, \operatorname{dist}((t,x),\Gamma)\}$, and

$$\left\| \frac{1}{\pi|\ln\varepsilon|} \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma) \right\|_{W^{-1,1}((-T_0,T_0)\times\mathbb{R}^N)} \le C|\ln\varepsilon|^{-1/2} \tag{1.21}$$

where $\mathcal{T}_\varepsilon(u)$ and $\mathcal{T}(\Gamma)$ denote the energy-momentum tensors for u and Γ (defined in (2.8) and (2.9) respectively). In all these conclusions, $C = C(T_0,\Gamma)$ is independent of independent of ε.

Remark 1.3. In Lemma 4 we give an explicit construction of initial data for which the conclusions of the theorem hold.

Remark 1.4. The proof shows that the solutions u from Theorem 2 have a defect near Γ; see (6.5) for a precise, if opaque, version of this assertion.

Remark 1.5. In both the above theorems, the constants C in the conclusions are at least exponential in T_0. That is, our proofs yield constants of the form $C = ae^{bT_0}$, where a, b themselves depend on Γ and T_0, and may blow up as $T_0 \nearrow T$.

Remark 1.6. Our results imply in particular that if we fix Γ as in either of the theorems above, then there exists a sequence (u_ε) of solutions of (1.1) such that the energy-momentum tensors $\delta_\varepsilon \mathcal{T}_\varepsilon(u_\varepsilon)$ converge weakly as measures in $(-T,T)\times\mathbb{R}^N$to $\mathcal{T}(\Gamma)$ if the scaling factor $\delta_\varepsilon = \delta_\varepsilon(k)$ is chosen correctly. This can be seen as a form of varifold convergence, analogous to results proved in [17, 3, 6, 7] for elliptic and parabolic equations, and discussed in the hyperbolic case in [5].

By providing quantitative estimates of $\|\delta_\varepsilon \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\|_{W^{-1,1}}$, however, our results are sharper than simple convergence results. This sharpening is significant, because convergence results strictly analogous to known results in the elliptic or parabolic cases *can fail* in the hyperbolic setting. That is, in our setting (but *not* for elliptic or parabolic problems) there exist sequences of solutions (u_ε) such that $\delta_\varepsilon \mathcal{T}_\varepsilon(u_\varepsilon)$ converges to a measure-valued tensor $\mathcal{T}$ supported on a codimension k set, but such that $\mathcal{T}$ is not the energy-momentum tensor for any timelike minimal surface Γ — in other words, $\mathcal{T}$ is not weakly stationary; see Section 1.4 below for explicit examples.

Remark 1.7. If we fix Γ and consider an associated sequence (u_ε) of solutions as found in Theorem 2 with $\varepsilon \to 0$, the uniform energy bounds (1.20) away from Γ combined with a classical argument of Shatah [36] imply that after passing to a subsequence, u_ε converges weakly in $H^1_{loc}([(-T,T)\times\mathbb{R}^N]\setminus\Gamma)$ to a wave map into S^1.

Remark 1.8. In both theorems, we ultimately rely on energy estimates in a frame that moves with Γ. These estimates (summarized in Theorem 3) assert more or less that energy remains concentrated around Γ on the same scale for $0 < t < T$ as it is at $t = 0$. The hypotheses for Theorem 3 are

- small energy away from Γ_0, see (2.31);
- a defect near Γ_0, see (2.36); and
- small energy, *given the presence of the defect*, near Γ_0, *in a frame that moves with* Γ, see (2.34) and (2.35).

Theorems 1, 2 follow from the special case of Theorem 3 in which the energy is, roughly speaking, as concentrated as possible around Γ_0. The fact that our results for $k = 1$ are considerably stronger than for $k = 2$ stems ultimately from the fact that when $k = 1$, for initial data that is nearly energetically optimal — essentially, (1.17), (1.18) or suitable small perturbations thereof — the energy is very sharply concentrated around Γ_0, whereas when $k = 2$, for the model initial data, energy is quite spread out. A more precise expression of this fact appears in (1.33).

1.3. **about the proofs.** A main issue in the analysis of (1.1) is to establish some kind of stability property of the moving defect — that is, the interface ($k = 1$) or "string" ($k = 2$). The relativistic invariance of the equation suggests that a defect should acquire extra energy

when it accelerates (and this is confirmed by our results, for example (1.16)), so we must rule out this extra energy as a potential source of instability. Our analysis starts from the observation that, for a solution that behaves as predicted in the formal arguments of [41, 32, 34] and others, a moving defect will always appear to be energetically optimal in the frame of reference of an observer who is moving with the defect.

1.3.1. *change of variables.* Motivated by this, we begin by rewriting the equation in a frame that follows the timelike minimal surface Γ, where the defect is expected to remain. In these variables, our task is to show that the solution is approximately constant, and we expect the defect to have some optimality property that we can exploit.

To define the change of variables, we start with a map H defined on $(-T,T)\times\mathbb{T}^n$ and parametrizing $\Gamma\subset(-T,T)\times\mathbb{R}^N$, and we extend H to a diffeomorphism ψ between, essentially, a neighborhood in $(-T,T)\times\mathbb{T}^n\times\mathbb{R}^k$ of $(-T,T)\times\mathbb{T}^n$ and a neighborhood of Γ in $\mathbb{R}^{1+N}$. We write ψ as a function of variables $y=(y^0,\dots,y^N)=(y^\tau,y^\nu)$, where $y^\tau=(y^0,\dots,y^n)$ are variables tangent to Γ, and $y^\nu=(y^{n+1},\dots,y^N)$ corresponds to directions normal to Γ. We always arrange that y^0 is a timelike coordinate and that all other coordinates are spacelike.

We will also write for example $D_\tau=(\partial_{y^0},\dots,\partial_{y^n})$ and $\nabla_\nu=(\partial_{y^{n+1}},\dots,\partial_{y^N})$. We generally write D for a space-time gradient, and ∇ for a gradient involving space-like variables only.

We then define $v=u\circ\psi$ on the domain of ψ. We find it convenient to write the equation satisfied by v (that is, equation (1.1), expressed in terms of the y variables) in the form

$$-\partial_{y^\alpha}(g^{\alpha\beta}\partial_{y^\beta}v)-b\cdot Dv+\frac{1}{\varepsilon^2}f(v)=0,\qquad b^\beta:=\frac{\partial_{y^\alpha}\sqrt{-g}}{\sqrt{-g}}g^{\alpha\beta}. \tag{1.22}$$

Here $G=(g_{\alpha\beta})$ is the expression in the y coordinates of the Minkowski metric, $(g^{\alpha\beta})=(g_{\alpha\beta})^{-1}$, $g=\det(g_{\alpha\beta})$, and we implicitly sum over repeated indices. The equation (1.22) enjoys certain useful properties, which are summarized in Proposition 1. Some of these follow from the specific form we choose for the map ψ, and the fact that Γ is a timelike minimal surface implies a key property of the coefficient b of the first-order term:

$$|b^\nu|\le C|y^\nu|\quad\text{ at } y=(y^\tau,y^\nu),\ \text{ for } b^\nu:=(b^{n+1},\dots,b^N). \tag{1.23}$$

We emphasize that the verification of (1.23) is *the only place* in our analysis where we explicitly invoke the fact that Γ is a minimal surface.

1.3.2. *energy estimates.* We now focus on v solving (1.22) on, say, $(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)$ for some $T_1<T$ and $\rho_0>0$, where $B_\nu(\rho_0):=\{y^\nu\in\mathbb{R}^k_\nu : |y^\nu|<\rho_0\}$. We will use the notation

$$e_{\varepsilon,\nu}(v):=\frac12|\nabla_\nu v|^2+\frac{1}{\varepsilon^2}F(v). \tag{1.24}$$

We introduce a scaling factor $\delta_\varepsilon=\delta_\varepsilon(k)$, see (2.1), chosen so that, heuristically,

$$\delta_\varepsilon\int_{\{y^\nu\in\mathbb{R}^k_\nu:|y^\nu|\le\rho\}} e_{\varepsilon,\nu}(v)(y^\tau,\cdot)dy^\nu\ge 1-o_\varepsilon(1),\quad\text{if } v(y^\tau,\cdot)\text{ has a defect near } y^\nu=0 \tag{1.25}$$

for every fixed ρ_1. This is made precise below. One of our goals is to show that if

$$\zeta_3(s) := \delta_\varepsilon \int_{\mathbb{T}^n \times W_\nu(s)} |D_\tau v|^2 + |y^\nu|^2 e_{\varepsilon,\nu}(v)\ dy^1 \cdots dy^N \Big|_{y^0=s}$$

is small when $s = 0$, say, then it remains small for a range of positive s. Here $W_\nu(s)$ is a neighborhood of the origin in $\mathbb{R}^k_\nu$ that may depend on the parameter s but will always contain a ball of fixed radius ρ. The smallness of ζ_3 is consistent with v having a large amount of energy, as long as it involves mostly the normal energy $e_{\varepsilon,\nu}(v)$ and is concentrated very near the codimension k surface $\{y^\nu = 0\}$.

Our strategy is to define some quantity $\zeta_1(s)$ such that

$$\zeta_1'(s) \le C\zeta_3(s) \tag{1.26}$$

and such that, under suitable additional assumptions,

$$\zeta_1(s) \ge c\zeta_3(s) - o_\varepsilon(1). \tag{1.27}$$

A main task will then be to show that these additional assumptions are preserved by the equation (1.22). If we can do this, we can easily use Grönwall's inequality to control the growth of ζ_3.

For the verification of (1.26), we define the approximately[1] conserved energy density

$$e_\varepsilon(v) = \frac{1}{2} a^{\alpha\beta} v_{y^\alpha} v_{y^\beta} + \frac{1}{\varepsilon^2} F(v) \tag{1.28}$$

where $a^{\alpha\beta}$ is a positive definite matrix related to $g^{\alpha\beta}$, see (2.16). (When we want to avoid any possibility of confusion, we will write $e_\varepsilon(v; G)$ for the above quantity, and $e_\varepsilon(u; \eta)$ for the energy defined in (1.10), with η denoting the expression in the original coordinates of the Minkowski metric.) We further define

$$\zeta_1(s) := \delta_\varepsilon \int_{\mathbb{T}^n \times W_\nu(s)} (1 + \kappa_2 |y^\nu|^2) e_\varepsilon(v)\ dy^1 \cdots dy^N \Big|_{y_0=s} - 1$$

where κ_2 is a constant to be selected in a moment. (It will turn out later that we can take $\kappa_2 = 1$ in the scalar case.) We hope to show that ζ_1 satisfies properties (1.26), (1.27) above.

Indeed, as long as the sets $W_\nu(s)$ are chosen to shrink rapidly enough, we show in Section 3 that the verification of (1.26) follows quite easily from the differential inequality

$$\frac{\partial}{\partial y^0} e_\varepsilon(v) \le \sum_{i=1}^N \frac{\partial}{\partial y^i} \varphi^i + C(|D_\tau v|^2 + |y^\nu|^2 |\nabla_\nu v|^2) \tag{1.29}$$

for some vector $\varphi = (\varphi^1, \ldots, \varphi^N)$. The differential inequality (1.29) in turn follows easily from (1.22), see Lemma 2. The key point in (1.29) is the factor $|y^\nu|^2$, which follows from (1.23) and hence from the fact that Γ is a minimal surface.

[1]The *exact* law expressing conservation of energy for (1.1) can of course be transposed to the y coordinates. As far as we know this is not useful for our problem, since it does not distinguish any good property of equation (1.22) resulting from the fact that the change of variables is built around a parametrization of a minimal surface.

To check (1.27), we first note that some of the good properties of (1.22) alluded to above imply that if κ_2 is chosen in a suitable way, see (2.23), then

$$(1+\kappa_2|y^\nu|^2)e_\varepsilon(v) \geq \ c|D_\tau v|^2 + (1+|y^\nu|^2)e_{\varepsilon,\nu}(v).$$

With this choice of κ_2,

$$\zeta_1(s) \geq c\zeta_3(s) + \int_{\mathbb{T}^n}\left(\delta_\varepsilon \int_{W_\nu(s)} e_{\varepsilon,\nu}(v)dy^\nu - 1\right) dy^1\cdots dy^n\Big|_{y_0=s}.$$

Thus, in view of the choice (1.25) of δ_ε, we can deduce (1.27) as long as we can check that $v(s,\cdot)$ has a defect confined near $\{(y^1,\dots,y^N)\in\mathbb{T}^n\times\mathbb{R}^k_\nu : y^\nu = 0\}$. (This is the additional assumption mentioned before (1.27).)

1.3.3. *a certain stability property.* We therefore introduce a "defect confinement functional" $\mathcal{D}: H^1(\mathbb{T}^n\times B_\nu(\rho_0))\to\mathbb{R}$ that is designed to have two properties. (This functional takes quite different forms in the two cases $k=1,2$ that we consider, see (3.1) and (5.1).) First, we require that

(1.30) $\mathcal{D}(v(s,\cdot))$ small $\Rightarrow$ "defect is confined" $\Rightarrow$ lower energy bounds $\Rightarrow$ (1.27) holds.

This sort of argument will eventually lead to an inequality of the simple form

$$\zeta_3(s) \leq C[\zeta_1(s)+\zeta_2(s)] + o_\varepsilon(1), \tag{1.31}$$

where

$$\zeta_2(s) = \mathcal{D}(v(s)).$$

Second, we need $\mathcal{D}$ to be such that

(1.32) changes in $\zeta_2(s)$ can be controlled by $\zeta_3(s)$.

Concrete versions of (1.30) and (1.32) are established in Section 3 for $k=1$ and Section 5 for $k=2$. Heuristically, (1.32) should hold because, if the defect strays away from $y^\nu=0$, then it should carry with it concentrations of energy that can be detected by ζ_3. In the case $k=1$, (1.32) will take the simple form $\zeta_2(s)\leq 2\zeta_2(0)+C\int_0^s\zeta_3(\sigma)d\sigma$. The corresponding estimate for $k=2$ is similar but slightly more complicated. In both cases, however, by combining (1.31) and a specific concrete version of (1.32) with (1.26), we obtain control over $\zeta_i(s)$ for $i=1,2,3$. This gives us a good deal of information about the behavior of v, from which all of our main conclusions are ultimately deduced.

One can view (1.31), (1.32) as a weak stability property of states w for which $\mathcal{D}(w)$ is small and for which the the inequality in (1.31) is almost saturated.

The difference in the strength of our conclusions in the cases $k=1,2$, discussed in Remark 1.8, stems from the fact that for optimal initial data,

$$\text{for } i=1,2,3, \quad \zeta_i(0)\approx\begin{cases}\varepsilon^2 & \text{when } k=1\\ |\ln\varepsilon|^{-1} & \text{for } k=2.\end{cases} \tag{1.33}$$

(See Lemma 4.) This reflects sharper energy concentration around $\{y^\nu=0\}$ in the case $k=1$.

1.3.4. *some other issues.* The change of variables that we employ is defined only in a neighborhood of Γ. We must therefore combine estimates of v near Γ with estimates of u away from Γ, and then iterate. We verify in Section 6 that this can be done in such a way as to genuinely yield estimates valid up to $(-T_0, T_0) \times \mathbb{R}^N$ for arbitrary $T_0 < T$.

Spacelike hypersurfaces of the form $\{y^0 = \text{constant}\}$ play a distinguished role in our argument, as it is along these surfaces that the defect structure is nearly energetically optimal for the solutions v that we consider. This near-optimality is manifested for example in the fact that inequality (1.31) is nearly saturated. In general our change of variables ψ^{-1} maps the hypersurface $\{(t,x) \in \mathbb{R}^{1+N} : t = 0\}$, on which we assume the data for the solution u of (1.1) is given, onto a hypersurface that is smooth and spacelike but otherwise can be quite arbitrary. So a certain amount of work is needed to obtain control of v on a suitable portion of some hypersurface $\{y^0 = \text{constant}\}$. This is done in Sections 4 and 5.3, and involves mainly technical adjustments to our basic energy estimates as outlined above. This means that we carry out our main energy estimates twice, once in a simpler form that can easily be iterated, and once to deal with complications caused by the geometry of the initial hypersurface in the transformed variables. This and the similarity between the cases $k = 1, 2$ leads to a certain amount of redundancy, which however enables us to present out argument first in a relatively simple setting, in Section 3; we believe this makes the main ideas easier to grasp.

The technical work of Section 4 could be avoided if we insisted on prescribing data only on spacelike hypersurfaces that have the form $\{y^0 = \text{constant}\}$ near Γ_0, but we feel that this would be unnecessarily restrictive.

Finally, we extract all the conclusions of the main theorems from control over quantities such as $\zeta_1, \zeta_2, \zeta_3$ above. This is done in Section 6. In the vector case, these arguments require a useful recent estimate of Kurzke and Spirn [23], without which we would not be able to establish the full energy-momentum tensor estimate (1.21).

1.4. **some examples.** It is well-known that the timelike minimal surface equation for $1+1$-dimensional surfaces in $\mathbb{R}^{1+N}$ is explicitly solvable for every $N \geq 2$. In particular, if $a : \mathbb{R} \to \mathbb{R}^N$ and $b : \mathbb{R} \to \mathbb{R}^N$ are smooth maps such that $|a'| = |b'| = 1$, then the function

$$X(s,t) := (t, x(s,t)), \qquad x(s,t) := \frac{1}{2}(a(s+t) + b(s-t))$$

parametrizes a surface that satisfies the timelike minimal surface equation wherever it is smooth. (See for example the exposition in [41], chapter 6.) This implies in particular that if $g : \mathbb{R} \to \mathbb{R}^k$ is any smooth function (where $k = N - 1$), then

$$\Gamma := \{(t, s, g(s-t)) : t, s \in \mathbb{R}\} \tag{1.34}$$

is a $1+1$-dimensional minimal surface in $\mathbb{R}^{1+N}$. For a timelike minimal surface Γ of this very simple form, it turns out that there are corresponding solutions of the nonlinear wave equation (1.1) that *exactly* follow Γ. Indeed, if $q : \mathbb{R}^k \to \mathbb{R}^k$ is any smooth solution of

$$-\Delta q + (q^2 - 1) q = 0$$

then writing $x \in \mathbb{R}^N = \mathbb{R}^{1+k}$ as $(x^1, x^\nu) \in \mathbb{R} \times \mathbb{R}^k$,

$$u(t,x) := q\Big(\frac{x^\nu - g(x^1 - t)}{\varepsilon}\Big) \tag{1.35}$$

solves (1.1) in all of $\mathbb{R}^{1+N}$.

In particular, consider a family of surfaces $(\Gamma^\varepsilon)_{\varepsilon\in 0,1]}$ of the form (1.34) associated with a sequence of smooth rapidly oscillating functions (g_ε) converging weakly in H^1, to a limiting function g_0. Although Γ^ε converges in the Hausdorff distance to the minimal surface Γ_0 associated via (1.34) with the function g_0, one can arrange the oscillation in such a way that $\mathcal{T}(\Gamma^\varepsilon)$ converges weakly to a limiting measure that is *not* equal to $\mathcal{T}(\Gamma_0)$. (This is a simple special case of the phenomenon known in the cosmology literature as "wiggly strings", see again [41] Chapter 6. Related issues are also discussed in [32].)

To illustrate this in detail, let us for simplicity assume that $k = 1$ and that $g_0 = 0$. One can check that if u_ε is the solution of the form (1.35) associated with g_ε, then (using notation defined in Section 2.3)

$$\mathcal{T}_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon^2} q'^2 \begin{pmatrix} 1+g_\varepsilon'^2 & -g_\varepsilon'^2 & g_\varepsilon' \\ g_\varepsilon'^2 & 1-g_\varepsilon'^2 & g_\varepsilon' \\ -g_\varepsilon' & g_\varepsilon' & 0 \end{pmatrix}, \qquad \text{and } \ \mathcal{T}(\Gamma_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathcal{H}^{1+1} \llcorner \Gamma_0.$$

From these it is easy to see that unless $g_\varepsilon \to g_0 = 0$ strongly in $H^1_{loc}(\mathbb{R})$, $\frac{\varepsilon}{\kappa_1}\mathcal{T}_\varepsilon(u_\varepsilon)$ converges to a limit that does not equal $\mathcal{T}(\Gamma_0)$. One can further check that this limit in general is not the energy-momentum tensor for any smooth string.

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2. NOTATION AND ASSUMPTIONS

2.1. **general notation.** We will write $B(\rho)$ to denote an open ball of radius ρ centered at the origin.

In order to emphasize the parallels between the two cases we consider, we will use the same notation for $k = 1, 2$, normally without indicating the dependence on k. For example we will write

$$\delta_\varepsilon := \begin{cases} \varepsilon/\kappa_1 & \text{when } k = 1, \text{ for } \kappa_1 \text{ defined in (1.11)} \\ (\pi|\ln\varepsilon|)^{-1} & \text{for } k = 2. \end{cases} \tag{2.1}$$

Similarly, $\mathcal{D}$ and $\mathcal{D}_\nu$ will have different meanings in the cases $k = 1, 2$, see (3.1)-(3.3) and (5.1)-(5.2).

Throughout this work we consider $1+n$-dimensional submanifolds in $1+N$-dimensional Minkowski space. We will always write $k = N - n$ for the codimension of the manifold. The same number k is also the dimension of the target space for the semilinear wave equation (1.1).

A parametric $(1 + n)$-dimensional submanifold Γ of $\mathbb{R}^{1+N}$ is a submanifold described as the image of a smooth map $H : U \to \mathbb{R}^{1+N}$ where U is an open subset of $\mathbb{R}^{1+n}$. We

will generally assume that this map H is injective. Given a map H parametrizing a surface Γ, we will often define a map $\psi : U \times (\text{small ball in } \mathbb{R}^k) \to \mathbb{R}^{1+N}$ that parametrizes a neighborhood of Γ and agrees with H on $U \times \{0\}$. In this situation, we will typically write points in $U \times \mathbb{R}^k \subset \mathbb{R}^{1+N}$ in the form

$$y = (y^\tau, y^\nu), \quad \text{with } y^\tau = (y^0, \dots, y^n) \in U \text{ and } y^\nu = (y^{n+1}, \dots, y^N) \in \mathbb{R}^k. \tag{2.2}$$

The superscripts stand for "tangential" and "normal" respectively. We will also sometimes use the alternate notation

$$y^\nu = (y^{\nu,1}, \dots, y^{\nu,k}) \tag{2.3}$$

for y^ν. We will always arrange that y^0 is a timelike coordinate, and we will often write $y^{\tau\prime} = (y^1, \dots y^n)$ and $y' := (y^{\tau\prime}, y^\nu)$, so that a "prime" denotes spatial variables only.

For notational consistency, we may sometimes write y^τ to denote a point $(y^0, \dots, y^n) \in U \subset \mathbb{R}^{1+n}$, even when there are no normal y^ν variables present. We may also write for example $\mathbb{R}^k_\nu$ to denote a copy of $\mathbb{R}^k$ that should be thought of as being in the normal y^ν variables, and we will write $B_\nu(\rho) := \{y^\nu \in \mathbb{R}^k_\nu : |y^\nu| < \rho\}$, where k should be clear from the context. We will generally write ∇ to denote the gradient in spatial directions only, and D to denote the spacetime gradient, so that $D = (\partial_t, \nabla)$. When using the notation (2.2), we will similarly write $D = (D_\tau, \nabla_\nu) = (\partial_{y^0}, \nabla_\tau, \nabla_\nu)$, where for example $\nabla_\nu = (\partial_{y^{n+1}}, \dots, \partial_{y^N})$.

We write $\eta = (\eta_{\alpha\beta}) = (\eta^{\alpha\beta})$ to denote the diagonal matrix $\operatorname{diag}(-1, 1, \dots, 1)$.

We normally follow the convention that Latin indices i, j, k run from 1 to N and Greek indices α, β, γ run from 0 to N, and we sum over repeated upper and lower indices. When summing implicitly over the (t, x) variables, we will identify x^0 with t.

2.2. **assumptions and notation related to timelike minimal surfaces.** A parametric submanifold is said to be *timelike* if $\gamma(DH) := \det(DH^T \eta\, DH) < 0$ at every point of U. The Minkowski area of a timelike parametric submanifold is defined to be

$$\mathcal{L}(H) := \int_U \sqrt{-\gamma} \tag{2.4}$$

A timelike submanifold $\Gamma = \text{Image}(H)$ is said to be a *timelike minimal surface* if H is a critical point of $\mathcal{L}$. (The terminology, although standard, is misleading, as a minimal surface Γ is in general not a minimizer or local minimizer of $\mathcal{L}$.)

Our main results all involve a timelike minimal surface Γ that is the image of a smooth, injective map $H : (-T, T) \times \mathbb{T}^n \to (-T, T) \times \mathbb{R}^N$ of the form

$$H(y^0, \dots, y^n) = \ (y^0, h(y^0, \dots, y^n)) \qquad \text{for some smooth } h : (-T, T) \times \mathbb{T}^n \to \mathbb{R}^N. \tag{2.5}$$

where $\mathbb{T}^n$ denotes the n-dimensional torus, thought of as the periodic unit cube (so that $\mathcal{H}^n(\mathbb{T}^n) = 1$). We will require that our parametrization satisfies[2]

$$H^T_{y_0}\, \eta\, H_{y_i} = h_{y_0} \cdot h_{y_i} = 0 \quad \text{ for } i > 0 \tag{2.6}$$

[2] Assumption (2.6) does not entail any loss of generality. Indeed, for H of the form (2.5), we can always achieve (2.6) by replacing h by a function $\tilde{h}$ of the form $\tilde{h}(y_0, \dots, y_n) = h(y_0, \Psi(y_0, \dots, y_n))$ for a suitable $\Psi : (-T, T) \times \mathbb{T}^n \to (-T, T) \times \mathbb{T}^n$. The suitable Ψ can be found by making the ansatz $\tilde{H}(y) = (y_0, \tilde{h}(y))$ for $\tilde{h}$, and substituting into (2.6). This yields an ordinary differential equation for Ψ that we can supplement with the initial conditions $\Psi(0, y') = y'$ and then solve by appealing to standard theory.

where here and throughout, we view H and h as column vectors. One can easily check that if Γ is a timelike parametric submanifold given as the image of a map H satisfying (2.5) and (2.6), then for any $T_1 < T$, there exists some $\alpha > 0$ such that

$$H_{y_0}^T \,\eta\, H_{y_0} = -1 + |h_{y_0}|^2 \le -\alpha, \qquad \nabla H^T \nabla H \ge \alpha I_n \qquad \text{for all } y^\tau \in (-T_1, T_1) \times \mathbb{T}^n. \tag{2.7}$$

2.3. **energy-momentum tensors.** Among other results, we establish a relationship between the energy-momentum tensors for a codimension k timelike Minkowski minimal surface in $\mathbb{R}^{1+N}$ and its counterpart for the the semilinear wave equation (1.1) for a function $\mathbb{R}^{1+N} \to \mathbb{R}^k$ with $0 < \varepsilon \ll 1$. We recall the definitions: if u solves (1.1), then $\mathcal{T}_\varepsilon(u)$ is defined to be the tensor whose components are

$$\mathcal{T}^\alpha_{\varepsilon,\beta}(u) := \delta^\alpha_\beta \left(\frac{1}{2} \eta^{\gamma\delta} u_{x^\gamma} \cdot u_{x^\delta} + \frac{1}{\varepsilon^2} F(u) \right) - \eta^{\alpha\gamma} u_{x^\gamma} \cdot u_{x^\beta}. \tag{2.8}$$

Here $(\eta^{\alpha\beta}) = \mathrm{diag}(-1, 1, \ldots, 1)$ as usual. (We deviate from convention in taking $\mathcal{T}_\varepsilon(u)$ and $\mathcal{T}(\Gamma)$ to be tensors of type $(1,1)$ rather than of type $(0,2)$; to recover the standard definition one must lower an index.)

And if Γ is a timelike minimal surface, then we define $\mathcal{T}(\Gamma)$ to be the tensor whose components are the signed measures

$$\mathcal{T}^\alpha_\beta(\Gamma)(A) := \int_A P^\alpha_\beta(t,x)\; d\lambda_\Gamma, \tag{2.9}$$

where λ_Γ denotes the Minkowski area density of Γ, and where $P(t,x) = (P^\alpha_\beta(t,x))$ is the matrix corresponding to Minkowski orthogonal projection onto $T_{(t,x)}\Gamma$, for λ_Γ a.e. $(t,x) \in \Gamma$. That is, if $H : U \subset \mathbb{R}^{1+n} \to \mathcal{U} \subset \mathbb{R}^{1+N}$ is a smooth injective map such that $\Gamma = H(U)$, then λ_Γ denotes the measure on $\mathcal{U}$ defined by

$$\int_{\mathbb{R}^{1+N}} f(x)\; d\lambda_\Gamma \;:=\; \int_U f(H(y^\tau)) \sqrt{-\gamma(y^\tau)}\; dy^\tau.$$

where as before $\gamma = \det(DH^T \,\eta\, DH)$. (It is easy to check that λ_Γ depends only on Γ.) And $P = P(t,x)$ is characterized by

$$P^\alpha_\beta v^\beta = v^\alpha \quad \text{ for } v \in T_{(t,x)}\Gamma, \qquad\qquad P^\alpha_\beta w^\beta = 0 \quad \text{ if } w^T \eta v = 0 \text{ for all } v \in T_{(t,x)}\Gamma.$$

For both models, the energy-momentum tensor may be obtained by considering variations of the relevant action functional with respect to suitable one-parameter families of diffeomorphisms. We recall this in some detail for $\mathcal{T}(\Gamma)$, as we will need to refer to this later:

Lemma 1. *Suppose that $H : U \subset \mathbb{R}^{1+n} \to \mathcal{U} \subset \mathbb{R}^{1+N}$ is a smooth injective map whose image $\Gamma := H(U)$ is a timelike surface. Given $\tau \in C^\infty_c(\mathcal{U}; \mathbb{R}^{1+N})$, define $\Phi_\sigma(x) := x + \sigma\tau(x)$. Then*

$$\frac{d}{d\sigma} \mathcal{L}(\Phi_\sigma \circ H)\Big|_{\sigma=0} \;=\; \int_{\mathcal{U}} \tau^\beta_{x^\alpha}(x)\;\; P^\alpha_\beta \, d\lambda_\Gamma \;=\; \int_{\mathcal{U}} \tau^\beta_{x^\alpha}(x)\;\; d\mathcal{T}^\alpha_\beta(\Gamma). \tag{2.10}$$

Note that (2.10) exactly parallels the well-known first variation formula in the Euclidean case, in which λ_Γ is replaced by the restriction to Γ of Hausdorff measure of the suitable

dimension, and P^α_β is replaced by orthogonal projection with respect to the Euclidean inner product.

Exactly parallel to (2.10), $\mathcal{T}_\varepsilon(u)$ arises from domain variations of the action functional, say $\mathcal{A}_\varepsilon$, whose Euler-Lagrange equation is (1.1), see for example [37] for the proof. Thus the results (1.15), (1.21) assert that the first variation of $\mathcal{A}_\varepsilon$ (with respect to domain variations) at the critical point u is close (in a weak topology, and after suitable rescaling) to the first variation of $\mathcal{L}$ at the associated timelike minimal surface Γ.

We present the standard calculation that leads to (2.10), since we will need it later:

Proof of Lemma 1. We will write $H_\sigma := \Phi_\sigma \circ H$, and

$$\gamma_{\sigma,ab} = H^T_{\sigma,y^a}\eta H_{\sigma,y^b} = H^\alpha_{\sigma,y^a}\eta_{\alpha\beta}H^\beta_{\sigma,y^b}, \qquad (\gamma^{ab}_\sigma) = (\gamma_{\sigma,ab})^{-1} \qquad \gamma_\sigma = \det(\gamma_{\sigma,ab}),$$

where indices a,b run from 0 to n and α,β as usual run from 0 to N. Using the fact that $\frac{d}{d\sigma}\gamma_\sigma = \gamma_\sigma\gamma^{ab}_\sigma\frac{d}{d\sigma}\gamma_{\sigma,ab}$ we find that

$$\begin{aligned}
\frac{d}{d\sigma}\mathcal{L}(H_\sigma)\Big|_{\sigma=0} \;=\; \frac{d}{d\sigma}\int_U\sqrt{-\gamma_\sigma}\Big|_{\sigma=0} &= \int_U(\tau^\beta\circ H)_{y^a}\eta_{\beta\delta}H^\delta_{y^b}\gamma^{ab}\sqrt{-\gamma}\;dy^\tau\\
&= \int_U(\tau^\beta_{x^\alpha}\circ H)H^\alpha_{y^a}\eta_{\beta\delta}H^\delta_{y^b}\gamma^{ab}\sqrt{-\gamma}\;dy^\tau\\
&= \int_{\mathcal{U}}\tau^\beta_{x^\alpha}(t,x)P^\alpha_\beta(t,x)\;d\lambda_\Gamma
\end{aligned}$$

where

$$P^\alpha_\beta(H(y^\tau)) := H^\alpha_{y^a}(y^\tau)\gamma^{ab}(y^\tau)H^\delta_{y^b}(y^\tau)\eta_{\delta\beta}.$$

Note that P^α_β is defined for λ_Γ a.e. (t,x), so the above integral makes sense. In order to complete the proof, we must check that $P^\alpha_\beta(t,x)$ is the orthogonal projection onto $T_{(t,x)}\Gamma$. To see this, first note that at any $y^\tau\in\mathbb{R}^{1+n}$,

$$(PH_{y^c})^\alpha = P^\alpha_\beta H^\beta_{y^c} = H^\alpha_{y^a}\gamma^{ab}H^\delta_{y^b}\eta_{\delta\beta}H^\beta_{y^c} = H^\alpha_{y^a}\gamma^{ab}\gamma_{bc} = H^\alpha_{y^a}\delta^a_c = H^\alpha_{y^c}.$$

Thus $PH_{y^c} = H_{y^c}$. And if v is orthogonal to H_{y^b} for all b, then

$$(Pv)^\alpha = P^\alpha_\beta v^\beta = H^\alpha_{y^a}\gamma^{ab}H^\delta_{y^b}\eta_{\delta\beta}v^\beta = 0$$

since the orthogonality of v means exactly that $H^\delta_{y^b}\eta_{\delta\beta}v^\beta = 0$ for every b. Since $T_{(t,x)}\Gamma$ at $(t,x) = H(y^\tau)$ is spanned by $\{H_{y^b}(y^\tau)\}^n_{b=0}$, the above calculations exactly state that $P(t,x)$ is the matrix corresponding to orthogonal projection onto $T_{(t,x)}\Gamma$. □

2.4. **change of variables.** We next define the change of variables that, as mentioned earlier, is the starting point of our argument. We will use the notation (2.2).

We assume as always that Γ is a smooth timelike minimal surface, given as the image[3] of a smooth injective map $H : (-T,T)\times\mathbb{T}^n \to \mathbb{R}^{1+N}$ satisfying (2.5), (2.7). For this section, we allow $k = N-n$ to be an arbitrary positive integer, since all the proofs for $k=2$ apply without change to $k\geq 3$. (The case $k=1$ is simpler.) Although we do not use them in this

[3]All the results of this section are local, so the topology of Γ, that is, the fact that H is defined on $(-T,T)\times\mathbb{T}^n$, is irrelevant here. But it is convenient to keep the same set-up as in the rest of the paper.

paper, the results for $k \geq 3$ may be useful for problems such as the dynamics of defects in certain nonabelian gauge theories.

First, we fix smooth maps $\bar\nu_i : (-T,T)\times\mathbb{T}^n \to \mathbb{R}^{1+N}$ for $i = 1,\ldots,k$ such that

$$\bar\nu_i^T\,\eta\,\bar\nu_j = \delta_{ij}, \qquad H^T_{y^\alpha}\,\eta\,\bar\nu_i = 0 \quad \text{in } (-T,T)\times\mathbb{T}^n \to \mathbb{R}^{1+N} \tag{2.11}$$

for all $i,j \in \{1,\ldots,k\}$ and $\alpha\in\{0,\ldots,n\}$. (Here and throughout the paper, we are thinking of $\bar\nu_i$ as a column vector.) This states that $\{\bar\nu_1(y^\tau),\ldots,\bar\nu_k(y^\tau)\}$ form an orthonormal basis for the normal space to Γ at $H(y^\tau)$, where words like "normal" and "orthonormal" are understood with respect to the Minkowski inner product and y^τ denotes a generic pont in $(-T,T)\times\mathbb{T}^n$. Note that when $k=1$, (2.11) determines $\bar\nu_1$ up to a sign, whereas for $k\geq 2$ there are rotational degrees of freedom that we have not specified (and will not specify).

Next, we define (using the notation (2.2))

$$\psi(y) \ := \ H(y^\tau) + \sum_{i=1}^k \bar\nu_i(y^\tau)\, y^{n+i}. \tag{2.12}$$

It is clear that $\psi(y^\tau,0) = H(y^\tau)$ for all $y^\tau\in(-T,T)\times\mathbb{T}^n$.

Recall that the statement of Theorems 1, 2 involve a number $T_0 < T$. We henceforth fix $T_1\in(T_0,T)$, and we let $\rho_0>0$ be so small that

(2.13)

$$\psi\left(\{-T_1\}\times\mathbb{T}^n\times B_\nu(\rho_0)\right) \subset\subset (-T,-T_0)\times\mathbb{R}^N, \qquad \psi\left(\{T_1\}\times\mathbb{T}^n\times B_\nu(\rho_0) \subset (T_0,T)\right)\times\mathbb{R}^N,$$

and

$$\psi \text{ is injective, with smooth inverse } \phi, \text{ on } (-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0). \tag{2.14}$$

The latter condition can be satisfied due to the inverse function theorem, as we will check below that $D\psi(y^\tau,0)$ is invertible for $y^\tau\in(-T_1,T_1)\times\mathbb{T}^n$. We next define

$$(g_{\alpha\beta})_{\alpha,\beta=0}^N = G := D\psi^T\ \eta\ D\psi \tag{2.15}$$

so that G represents the Minkowski metric in the y coordinates. We further define $g := \det G$ and $(g^{\alpha\beta})_{\alpha,\beta=0}^N := G^{-1}$, and we finally define $(a^{\alpha\beta})_{\alpha,\beta=0}^N$ by

$$a^{ij} = g^{ij} \ \text{ if } i,j\geq 1, \quad a^{00} = -g^{00}, \quad a^{i0} = a^{0j} = 0 \ \text{ for } i,j = 1,\ldots,N. \tag{2.16}$$

When we write (1.1) in terms of the y coordinates as in (1.22), $(g^{\alpha\beta})$ and g appear in the coefficients, and $(a^{\alpha\beta})$ appears in a natural associated energy density $e_\varepsilon(v) = e_\varepsilon(v;G)$, defined in (1.28). We summarize properties of g and $(g^{\alpha\beta})$ that we will use:

Proposition 1. *Let $\psi, g, (g^{\alpha\beta})$ be the functions on $(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)$ defined above. Then, after taking ρ_0 smaller if necessary, there exist positive constants $c\leq C$ such that*

$$\|g^{\alpha\beta}\|_{W^{1,\infty}} \leq C \qquad g^{\alpha\beta}_{y^0}\xi_\alpha\xi_\beta \leq C(|\xi_\tau|^2 + |y^\nu|^2\,|\xi_\nu|^2) \tag{2.17}$$

$$\frac{\partial_{y^\alpha}\sqrt{-g}}{\sqrt{-g}}\, g^{\alpha\beta}\xi_\beta\xi_0 \leq C(|\xi_\tau|^2 + |y^\nu|^2\,|\xi_\nu|^2), \tag{2.18}$$

$$|g^{\alpha\beta}\xi_\beta| \leq C(|\xi_\tau| + |y^\nu||\xi_\nu|) \qquad \textit{if } \alpha\leq n, \tag{2.19}$$

and

$$c|\xi_\tau|^2 + (1 - C|y^\nu|^2)|\xi_\nu|^2 \ \le \ a^{\alpha\beta}(y)\xi_\alpha\xi_\beta \ \le \ C|\xi_\tau|^2 + (1 + C|y^\nu|^2)|\xi_\nu|^2 \tag{2.20}$$

for all $y = (y^\tau, y^\nu) \in (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$ *and* $\xi = (\xi_\tau, \xi_\nu) \in \mathbb{R}^{1+N} \cong \mathbb{R}^{1+n} \times \mathbb{R}^k$. *In addition,*

$$\psi^0_{y^0} \ge c \quad \text{in } (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0). \tag{2.21}$$

We will use the notation

$$\mathcal{N} := \psi\left((-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)\right) \cap [(-T_0, T_0) \times \mathbb{R}^N] \tag{2.22}$$

For future use, it is convenient to fix a constant $\kappa_2 \ge 1$ such that

$$(1 + \kappa_2|y^\nu|^2)e_\varepsilon(v) \ge \frac{\lambda}{2}|D_\tau v|^2 + (1 + |y^\nu|^2)e_{\varepsilon,\nu}(v) \tag{2.23}$$

everywhere in $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$, for all $v \in H^1$, where $\varepsilon_{\varepsilon,\nu}$ was defined in (1.24). This is possible due to (2.20).

When Γ is a hypersurface, we have slightly better behavior:

Proposition 2. *Suppose that* $k = 1$, *and let* $\psi, g, (g^{\alpha\beta})$ *be as defined above. Then, after taking* ρ_0 *smaller if necessary,*

$$g^{\alpha N} = g^{N\alpha} = \begin{cases} 1 & \text{if } \alpha = N \\ 0 & \text{if not.} \end{cases} \tag{2.24}$$

$$\lambda|\xi_\tau|^2 + |\xi_\nu|^2 \ \le \ a^{\alpha\beta}(y)\xi_\alpha\xi_\beta \ \le \ \Lambda|\xi_\tau|^2 + |\xi_\nu|^2. \tag{2.25}$$

everywhere in $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$.

Conclusion (2.25) is not essential but will allow us to simplify our notation, for example by taking $\kappa_2 = 1$ in (2.23) and everywhere else that this constant occurs (for $k = 1$).

We defer the proofs of Propositions 1 and 2 to an Appendix, see Section 7.

For a solution $u : \mathbb{R}^{1+N} \to \mathbb{R}^k$ of (1.1), we will define $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}^k$ by $v = u \circ \psi$. Then v satisfies

$$\Box_G v + \frac{1}{\varepsilon^2} f(v) = 0 \tag{2.26}$$

on its domain. Here

$$\Box_G v = -\frac{1}{\sqrt{-g}} \partial_{y^\alpha}(\sqrt{-g} g^{\alpha\beta} \partial_{y^\beta} v).$$

As noted earlier, we find it convenient to write (2.26) in the form (1.22). We now derive a key differential inequality for the energy density $e_\varepsilon(v)$ from (1.28).

Lemma 2. *Suppose that* $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}^k$ *is a smooth solution of* (2.26), *with coefficients satsfiying* (2.17). *Then*

$$\frac{\partial}{\partial y^0} e_\varepsilon(v) \le C(|D_\tau v|^2 + |y^\nu|^2 \ |\nabla_\nu v|^2) + \nabla \cdot \varphi \tag{2.27}$$

with

$$\varphi := (\varphi^1, \ldots, \varphi^N), \qquad \varphi^i := \ g^{i\alpha} v_{y^\alpha} \cdot v_{y^0}. \tag{2.28}$$

Proof. Multiply (1.22) by v_{y^0} and rewrite to find that

$$-\partial_{y^\alpha}(g^{\alpha\beta} v_{y^\beta} \cdot v_{y^0}) + g^{\alpha\beta} v_{y^\beta} \cdot v_{y^0 y^\alpha} + \frac{1}{\varepsilon^2} F(v)_{y^0} = -(b \cdot Dv) \cdot v_{y^0}.$$

We rewrite $g^{\alpha\beta} v_{y^\beta} \cdot v_{y^0 y^\alpha}$ as $\frac{1}{2}\partial_{y^0}(g^{\alpha\beta} v_{y^\beta} \cdot v_{y^\alpha}) - \frac{1}{2} g^{\alpha\beta}_{y^0} v_{y^\beta} \cdot v_{y^\alpha}$. Gathering all the terms of the form $\partial_{y^0}[\ldots]$ on the left-hand side, we find that

$$\partial_{y^0}\left[-g^{0\beta} v_{y^\beta} \cdot v_{y^0} + \frac{1}{2} g^{\alpha\beta} v_{y^\alpha} \cdot v_{y^\beta} + \frac{1}{\varepsilon^2} F(v)\right] = \partial_{y^i}(g^{i\beta} v_{y^\beta} \cdot v_{y^0}) - (b \cdot Dv) v_{y^0} + \frac{1}{2} g^{\alpha\beta}_{y^0} v_{y^\beta} \cdot v_{y^\alpha}.$$

The definition (2.16) of $a^{\alpha\beta}$ implies that left-hand side is just $\partial_{y^0} e_\varepsilon(v)$. To complete the proof, we use (2.17) and (2.18) to check that the non-divergence terms on the right-hand side are bounded by $C(|D_\tau v|^2 + (y^\nu)^2 |\nabla_\nu v|^2)$. $\square$

As an easy consequence of Proposition 2, we obtain a quite explicit description of the signed Minkowski distance function defined by the eikonal equation (1.12) in the case $k = 1$.

Corollary 1. *Assume that $k = 1$ and define ψ as above, and let $\phi = (\phi^0, \ldots, \phi^N)$ denote the inverse of ψ. Then ϕ^N solves the eikonal equation* (1.12) *on Image*(ψ).

In particular, the Corollary shows that it makes sense to speak of the signed distance function in the set $\mathcal{N}$ defined in (2.22).

Proof. Fix a point in the image of ψ, say $(t, x) = \psi(y)$. Then since $\eta = \eta^{-1}$,

$$(g^{\alpha\beta})(y) = [D\psi^T(y) \ \eta \ D\psi(y)]^{-1} = (D\psi)^{-1}(y) \ \eta \ (D\psi)^{-T}(y) = D\phi(t, x) \ \eta \ D\phi^T(t, x).$$

Thus, according to (2.24),

$$1 = g^{NN}(y) = -(\phi^N_t)^2 + |\nabla \phi^N|^2,$$

so that (1.12) holds. And it is clear that $\phi^N(t, x) = 0$ for $(t, x) \in \Gamma$. $\square$

In fact the curves $s \mapsto H(y^\tau) + s\nu(y^\tau) = \psi(y^\tau, s)$ are exactly characteristic curves for the eikonal equation (1.12).

The eikonal equation (1.12) determines the distance function d only up to a sign; we will always choose to identify d with ϕ^N (so that our choice of a sign is ultimately determined by our choice of the sign for the unit normal ν.) Then it follows that

$$d(\psi(y)) = y^N \quad \text{for } y \in (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0), \tag{2.29}$$

2.5. **initial data.** In this section we describe our general assumptions on the initial data.

We will eventually combine estimates for $v = u \circ \psi$ on $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$, which we use control to the behavior of u near Γ, with standard energy estimates for (1.1) away from Γ. We start by making a number of smallness assumptions, in all of which a parameter ζ_0 appears. We will prove below that one can find data for which $\zeta_0 \approx \varepsilon^2$ when $k = 1$, and $\zeta_0 \approx |\ln \varepsilon|^{-1}$ when $k = 2$. Although we omit the proof, it is in fact true that one cannot find data satisfying our assumptions with $\zeta_0 \ll \varepsilon^2$ (for $k = 1$) or $\zeta_0 \ll |\ln \varepsilon|^{-1}$. We therefore will assume that

$$\zeta_0 \geq \varepsilon^2 \quad \text{if } k = 1, \qquad\qquad \zeta_0 \geq |\ln \varepsilon|^{-1} \quad \text{if } k = 2. \tag{2.30}$$

This is convenient, as it will enable us to absorb small error terms into expressions of the form $C\zeta_0$.

Our first assumption is that the energy is small away from Γ_0:

$$\delta_\varepsilon \int_{\{x \in \mathbb{R}^N : (0,x) \notin \text{ image}(\psi)\}} e_\varepsilon(u) dx \bigg|_{t=0} \leq \zeta_0 \tag{2.31}$$

where $e_\varepsilon(u) = e_\varepsilon(u; \eta)$ is defined in (1.10) and $\delta_\varepsilon = \delta_\varepsilon(k)$ is defined in (2.1).

Near Γ_0, it is convenient to state our assumptions in terms of $v = u \circ \psi$. Note that initial data for u at $t = 0$ corresponds to data for v on a hypersurface that does *not* in general have the form $\{y_0 = \text{const}\}$. This hypersurface is described in the following

Lemma 3. *There exists a Lipschitz function $b : \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}$ such that for $y = (y_0, y') \in (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$,*

$$\psi(y_0, y') \in \{0\} \times \mathbb{R}^N \text{ if and only if } y_0 = b(y'). \tag{2.32}$$

Moreover, $\|\nabla b\|_\infty \leq C$.

Proof. Fix $y' \in \mathbb{T}^n \times B_\nu(\rho_0)$, and for $s \in (-T_1, T_1)$, let $y(s) := (s, y')$ and let $X(s) = \psi(y(s)) \in \mathbb{R}^{1+N}$. To prove that $\psi^{-1}(\{0\} \times \mathbb{R}^N)$ is the graph of a function, we need to show that $y(s)$ intersects $\psi^{-1}(\{0\} \times \mathbb{R}^N)$ exactly once, or equivalently, that $X(s)$ intersects $\{0\} \times \mathbb{R}^N$ for exactly one value of s. To prove this, note that the definition of G and (2.7) imply that, after taking ρ_0 smaller if necessary,

$$X'(s)^T \eta X'(s) = y'(s)^T G(y(s)) y'(s) = g_{00}(y(s)) < 0$$

for every s. Thus $s \mapsto X(s) = (X^0(s), X'(s))$ is a timelike curve, from which the claim is obvious. It follows that there exists a function b satisfying (2.32). Then by differentiating the identity $\psi^0(b(y'), y') = 0$, we find that $\psi^0_{y^0}(b(y'), y')\nabla b(y') + \nabla \psi^0(b(y'), y') = 0$. We know from (2.21) that $\psi^0_{y^0}$ is bounded away from 0, and this together with the smoothness of ψ^0 implies that $\|\nabla b\|_\infty \leq C$. $\square$

Using the lemma, we define

$$v_0(y') := v(b(y'), y') \text{ for } y' \in \mathbb{T}^n \times B_\nu(\rho_0). \tag{2.33}$$

Our next assumptions specify that the energy near Γ_0 is small, in the frame that moves with Γ:

$$\delta_\varepsilon \int_{\mathbb{T}^n\times B_\nu(\rho_0)} (1+\kappa_2|y^\nu|^2)e_\varepsilon(v_0;G)dy' - 1 \ \le\ \zeta_0, \tag{2.34}$$

$$\delta_\varepsilon \int_{\mathbb{T}^n\times B_\nu(\rho_0)} (|v_{y^0}|^2 + |v_{y^0}|\ |\nabla_\nu v_0|)(b(y'),y')\,dy' \ \le\ \zeta_0. \tag{2.35}$$

Finally, using notation discussed in the Introduction and defined in (3.1) for $k=1$ and (5.1), (5.3) for the case $k=2$, we require that

$$\mathcal{D}(v_0;\rho_0)\le\zeta_0. \tag{2.36}$$

This specifies that the initial profile possesses a defect — that is, an interface or vortex — near Γ_0.

Note that conditions (2.31), (2.34)-(2.36) are always satisfied if we define ζ_0 to be the maximum of the left-hand sides of these inequalities. The smallest possible values of ζ_0 depend on k and, as mentioned earlier, account for the fact that our conclusions for $k=1$ are stronger than for $k=2$.

Lemma 4. *In the scalar ($k=1$) case, there exists initial data $(u,u_t)|_{t=0}\in\dot H^1\times L^2(\mathbb{R}^N)$ for* (1.1) *satisfying conditions* (2.31) – (2.36) *with $\zeta_0=C\varepsilon^2$, and such that*

$$\int_{\mathcal{N}_0}\left(u(0,x)-q(\frac{d(0,x)}{\varepsilon})\right)^2\le C\varepsilon,\quad \textit{where}\ \ \mathcal{N}_0=\{x\in\mathbb{R}^N:(0,x)\in\mathcal{N}\}. \tag{2.37}$$

And in the vector ($k=2$) case, there exists initial data $(u,u_t)|_{t=0}\in\dot H^1\times L^2(\mathbb{R}^N;\mathbb{R}^2)$ for (1.1) *satisfying conditions* (2.31) – (2.36) *with $\zeta_0=C|\ln\varepsilon|^{-1}$.*

Although we do not prove it, these scalings for ζ_0 are in fact optimal.

Proof. In both cases $k=1,2$, we define a function U in Image(ψ) such that

$$U\circ\psi=\tilde q(\frac{y^\nu}{\varepsilon}) \tag{2.38}$$

where $\tilde q:\mathbb{R}^k\to\mathbb{R}^k$ is a nearly-optimal profile. We then require that

$$u(0,x)=U(0,x)\text{ and }u_t(0,x)=U_t(0,x)\text{ in }\mathcal{N}_0 \tag{2.39}$$

and we verify (2.34) - (2.36). (Note that (2.29) then implies that $u(x,0)=\tilde q(\frac{d}{\varepsilon})$ when $k=1$, which will make (2.37) obvious.) Finally, we argue that $u(0,\cdot)$ can be extended to $\mathbb{R}^N\setminus\mathcal{N}_0$ such that (2.31) holds.

k=1: By integrating the equation (1.5) solved by q and using the boundary conditions at $\pm\infty$, one finds that $q'=\sqrt{2F(q)}$, and hence that

$$\int_{\mathbb{R}}\frac12 q'^2+F(q)dx\ =\ \int_{\mathbb{R}}\sqrt{2F(q)}q'(s)ds=\int_{-1}^1\sqrt{2F(s)}\,ds\ =\ \kappa_1. \tag{2.40}$$

Using (1.5) and (1.9), standard ODE arguments show that

$$|q'(s)|+|q(s)-\operatorname{sign}(s)|\le Ce^{-c|s|}\qquad\text{for all }s.$$

for suitable constants. It follows that, given $\varepsilon > 0$, we can find a function $\tilde q$ such that $\tilde q(\frac{s}{\varepsilon}) = q(\frac{s}{\varepsilon})$ if $|s| < \frac12\rho_0$, and

$$\tilde q(\frac{s}{\varepsilon}) = q(\frac{s}{\varepsilon}) \text{ if } |s| < \frac13\rho_0, \qquad \tilde q(\frac{s}{\varepsilon}) = \mathrm{sign}(s) \text{ if } |s| > \frac23\rho_0, \qquad \|\tilde q - q\|_{W^{1,\infty}} \le Ce^{-c/\varepsilon}$$

and

$$\kappa_1 < \int_{-\rho_0/\varepsilon}^{\rho_0/\varepsilon} \frac12 \tilde q'^2 + F(\tilde q)dx \le \kappa_1 + Ce^{-c/\varepsilon}. \tag{2.41}$$

Now define U as in (2.38), and define $u|_{t=0}$ near Γ_0 by (2.39). Then by construction v_0 as defined in (2.33) is given by $v_0(y) = \tilde q(y^N/\varepsilon)$, and $v_{y_0} = 0$. The latter fact immediately implies that (2.35) holds, and (2.31), (2.35) are easily verified. For example, the explicit form of v_0 and (2.25) imply that $e_\varepsilon(v_0; G) = \frac12 \tilde q'^2(\frac{y^\nu}{\varepsilon}) + \varepsilon^{-2}F(\tilde q(\frac{y^\nu}{\varepsilon}))$. Then recalling that $\delta_\varepsilon = \frac{\varepsilon}{\kappa_1}$, we infer from (2.41) and a change of variables $y^N/\varepsilon \mapsto y^N$ that

$$\delta_\varepsilon \int_{\mathbb{T}^n\times B_\nu(\rho_0)} (1+\kappa_2|y^\nu|^2)e_\varepsilon(v_0;G)dy' - 1 \ \le C\varepsilon^2 \int_{-\rho_0/\varepsilon}^{\rho_0/\varepsilon} [\frac{\tilde q'^2}{2} + F(\tilde q)](y^N)^2 dy^N + Ce^{-c/\varepsilon}.$$

The exponential decay of q implies that $\int_R [\frac12 \tilde q'^2 + F(\tilde q)](y^N)^2dy^N \le C$ independent of ε, and (2.34) follows with $\zeta_0 = C\varepsilon^2$. The verifications of (2.35) and (2.36) are similar and a little easier.

Finally, on $\mathbb{R}^N \setminus \mathcal{N}_0$, we set $u_t(0,\cdot) \equiv 0$, and we require that $u(0,\cdot) = \pm 1$ and that u is continuous (hence smooth) across $\partial\mathcal{N}_0$. This can be done, since $\mathbb{R}^N \setminus \Gamma_0$ consists of two components, one of which meets $\mathcal{N}_0$ where $d = \rho_0$ (and hence $u = 1$), and the other where $d = -\rho_0$. (Here we have used the fact that ρ_0 is sufficiently small, see (2.13).)

k=2: In this case we may define $\tilde q(s) = s\min\{1, \frac{1}{|s|}\}$ for $s \in \mathbb{R}^2$, and we go on to make the definitions (2.38), (2.39) as above, so that $v_0(y) = \tilde q(y^\nu/\varepsilon)$. Then an easy calculation shows that

$$\frac{1}{\pi|\ln\varepsilon|}\int_{B_\nu(\rho_0/\varepsilon)} \frac12|\nabla\tilde q|^2 + F(\tilde q)ds \le 1 + C|\ln\varepsilon|^{-1}.$$

This plays a role analogous to (2.41) above and allows us to verify along the above lines (but using (2.20) in place of (2.25)) that (2.34) holds with $\zeta_0 = C|\ln\varepsilon|^{-1}$. As above, (2.35) follows from the fact that $v_{y^0}(b(y'), y') = 0$ in $\mathbb{T}^n \times B_\nu(\rho_0)$. One can check (2.36) directly from the definitions (see Section 5), noting that $J_\nu v_0(y^\tau, y^\nu) = \begin{cases} \varepsilon^{-2} & \text{if } |y^\nu| < \varepsilon, \\ 0 & \text{if } |y^\nu| > \varepsilon. \end{cases}$

It remains to show that $u_0 = U(0,\cdot)$, as defined in $\mathcal{N}_0$ by (2.39), can be extended to a function in $H^1(\mathbb{R}^N)$ satisfying (2.31). It is clear that we can extend u_0 by a finite-energy map in a neighborhood $\mathcal{V}$ of $\mathcal{N}_0$. Next we point out that since Γ_0 is a smooth, compact, oriented codimension 2 submanifold without boundary of $\mathbb{R}^N$, results in [1] imply that we may find a function $w \in H^1_{loc}(\mathbb{R}^N \setminus \Gamma; \mathbb{C})$ with $\int_{\mathbb{R}^N\setminus\mathcal{N}_0} |\nabla w|^2 < \infty$, such that $|w| = 1$ a.e., and in addition such that $Jw = J(\frac{u_0}{|u_0|})$ in Γ_0, where $J(\cdots)$ denotes the distributional Jacobian of $(\cdots)$. This implies that there exists a real-valued function $\theta \in H^1_{loc}(\mathcal{V} \setminus \Gamma_0; \mathbb{R})$ such that

$u_0 = |u_0|we^{i\theta}$ in $\mathcal{V}$. Thus we define $u(0,\cdot)$ globally in $\mathbb{R}^N$ by setting

$$u(0,\cdot) = \begin{cases} |u_0|we^{i\chi\theta} & \text{in } \mathcal{V} \\ w & \text{in } \mathbb{R}^N \setminus \mathcal{V} \end{cases}$$

where $\chi \in C_c^\infty(\mathcal{V})$ and $\chi \equiv 1$ in $\mathcal{N}_0$. We may set $u_t(0,x) = 0$ outside of $\mathcal{N}_0$. $\square$

3. Basic energy estimates, $k = 1$

The main result of this section, Proposition 3 below, contains the simplest case of our main estimate.

In this section and the next, we restrict our attention to the case $k = 1$, so that[4] $N = n+1$, $y^\nu = y^N \in \mathbb{R}$, and $\nabla_\nu = \partial_N$. Thus in this section, $B_\nu(\rho)$ denotes the interval $(-\rho, \rho)$ along the y^N axis. We also follow other conventions for $k = 1$, so that for example $\delta_\varepsilon = \frac{\varepsilon}{\kappa_1}$, see (2.1).

Throughout this section, we let ψ denote the change of variables from Section 2.4, in the case $k = 1$. We also use the notation $g, g_{\alpha\beta}, g^{\alpha\beta}$ etc from the previous section.

In the Introduction we discussed a "defect confinement" functional $\mathcal{D}$. In the case $k = 1$ we define it to be

$$\mathcal{D}(v;\rho) := \int_{\mathbb{T}^n \times B_\nu(\rho)} |y^\nu| \, |v - \text{sign}(y^\nu)|^2 dy' \tag{3.1}$$

for $v : \mathbb{T}^n \times B_\nu(\rho) \to \mathbb{R}$. We will also write

$$\mathcal{D}(v;\rho) = \int_{\mathbb{T}^n} \mathcal{D}_\nu(v(y^{\tau\prime}); \rho) dy^{\tau\prime}. \tag{3.2}$$

where $v(y^{\tau\prime})(y^\nu) = v(y^{\tau\prime}, y^\nu)$ and

$$\mathcal{D}_\nu(w;\rho) := \int_{B_\nu(\rho)} |y^\nu| \, |w - \text{sign}(y^\nu)|^2 dy^\nu \qquad \text{for } w : B_\nu(\rho) \to \mathbb{R}. \tag{3.3}$$

Let c_* be a constant such that

$$|g^{N\alpha}(y)\xi_\alpha\xi_0| \ = \ |a^{N\alpha}(y)\xi_\alpha\xi_0| \ \le \ \frac{1}{2}c_* \ a^{\alpha\beta}\xi_\alpha\xi_\beta \tag{3.4}$$

for all $\xi \in \mathbb{R}^{1+N}$ and $y \in (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$.

The main result of this section is

Proposition 3. *Let $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}$ satisfy (2.26), where $f = F'$ and F satisfies (1.9). Recalling that $\delta_\varepsilon = \frac{\varepsilon}{\kappa_1}$, where κ_1 is defined in (1.11), assume that there exist some $s_1 \in (-T_1, T_1)$, $\rho_1 \in (0, \rho_0)$, and $\zeta_0 \ge \varepsilon^2$ such that*

$$\delta_\varepsilon \int_{\{s_1\}\times\mathbb{T}^n\times B_\nu(\rho_1)} (1 + (y^\nu)^2) \, e_\varepsilon(v) dy' - 1 \ \le \zeta_0 \tag{3.5}$$

$$\mathcal{D}(v(s_1), \rho_1/2) \le \zeta_0. \tag{3.6}$$

[4]Although here there is not much point in writing y^ν and ∇_ν instead of y^N and ∂_N, this notation will prove useful when we consider the vector case, and we use it here to emphasize the parallels.

Then there exists a constant C, independent of v and of $\varepsilon \in (0,1]$, such that

$$\delta_\varepsilon \int_{\{s_1+s\}\times\mathbb{T}^n\times B_\nu(\rho_1-c_*s)} |D_\tau v|^2 + (y^\nu)^2\left[|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2}F(v)\right]\ dy' \le C\zeta_0$$

$$\delta_\varepsilon \int_{\{s_1+s\}\times\mathbb{T}^n\times B_\nu(\rho_1-c_*s)} (1+(y^\nu)^2)e_\varepsilon(v)\ dy' \ -\ 1\ \le C\zeta_0$$

and

$$\mathcal{D}(s_1+s;\rho_1/2) \le C\zeta_0$$

for all $s \in [0, \rho_1/2c_*]$ such that $s_1 + s < T_1$.

Our first Lemma will be needed to establish requirement (1.30) as discussed in the Introduction. In the statement and proof we take all the y^τ variables to be frozen, and we consider a function v of a single real variable y^ν.

Lemma 5. *Let $B_\nu(\rho) := (-\rho,\rho) \subset \mathbb{R}_\nu$ be an interval as above. Then there exists a constant $\kappa_3 = \kappa_3(\rho)$ such that if $v \in H^1(B_\nu(\rho))$ and if*

$$\mathcal{D}_\nu(v;\rho)\ \le\ \kappa_3 \tag{3.7}$$

then

$$\delta_\varepsilon \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(v)\ dy^\nu \ge 1 - Ce^{-C/\varepsilon}. \tag{3.8}$$

Moreover, there exists a constant $\kappa_4 = \kappa_4(\rho)$ such that if (3.7) holds and if

$$\delta_\varepsilon \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(v)\ dy^\nu \le 1 + \zeta_0 \qquad \textit{for some } \zeta_0 \in (0,\kappa_4), \tag{3.9}$$

then

$$\int_{B_\nu(\rho)} \left|\frac{\varepsilon}{2}v_{y^\nu}^2 - \frac{1}{\varepsilon}F(v)\right| dy^\nu \le C(\sqrt{\zeta_0} + e^{-c/\varepsilon}). \tag{3.10}$$

The proof of Proposition 3 uses only the first conclusion (3.8) of this lemma. The other conclusion (3.10) is used in the proof of Theorem 3, when we deduce control over the full energy-momentum tensor from simpler energy estimates like those of Proposition 3.

Proof of Lemma 5. **1**. Note that (3.8) is obvious if (3.9) fails, so it suffices to show that if (3.7) and (3.9) hold, then both conclusions (3.8), (3.10) follow.

First, we define $Q(s) := \int_0^s \sqrt{2F(\sigma)}d\sigma$, and for any function $w \in H^1(B_\nu(\rho))$, we estimate

$$\varepsilon e_{\varepsilon,\nu}(w) = \frac{\varepsilon}{2}w_{y^\nu}^2 + \frac{1}{\varepsilon}F(w) \ge \sqrt{2F(w)}|w_{y^\nu}| = |\partial_{y^\nu}(Q\circ w)|.$$

Thus since $\delta_\varepsilon = \frac{\varepsilon}{\kappa_1}$,

$$\delta_\varepsilon \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(w)\ \ge \frac{1}{\kappa_1}\int_{B_\nu(\rho)} |\partial_{y^\nu}(Q\circ w)|, \tag{3.11}$$

and for any w, to obtain lower bounds for the left-hand side, it suffices to show that $y^\nu \mapsto Q(w(y^\nu))$ has large total variation on $B_\nu(\rho) = (-\rho,\rho)$.

2. Next, fix $\alpha > 0$ so that $F' = f$ is decreasing on $(1-\alpha, 1)$; this is possible as F is C^2 and attains its minimum at 1 with $F''(1) > 0$.

Let $v^+ := \sup_{y^\nu \in (\frac{\rho}{4}, \frac{3\rho}{4})} v(y^\nu)$. If $v^+ \le 1$, then (3.7) implies that

$$\kappa_3 \ge \int_{\rho/4}^{3\rho/4} y^\nu |1 - v(y^\nu)|^2 dy^\nu \ge C\rho^2 (1 - v^+)^2.$$

Thus by choosing κ_3 small enough we can arrange that $v^+ \ge 1 - \theta\alpha$ for some $\theta \in (0, \frac{1}{2})$ to be chosen below. It then follows by the same argument that $v^- := \inf_{y^\nu \in (-\frac{3\rho}{4}, -\frac{\rho}{4})} v(y^\nu) \le -1 + \theta\alpha$.

3. We next claim that, once κ_3 and κ_4 are fixed in a suitable way, our hypotheses imply that

$$v \ge 1 - \alpha \quad \text{in } (\frac{3\rho}{4}, \rho) \qquad \text{and} \quad v \le -1 + \alpha \quad \text{in } (-\rho, -\frac{3\rho}{4}). \tag{3.12}$$

This follows from (3.11) and Step 2 — the latter implies lower bounds on the total variation of $Q \circ w$ if (3.12) fails, and these lower bounds can be made to contradict (3.11) and (3.9).

In more detail, let us suppose toward a contradiction that the first inequality in (3.12) fails. Then using Step 2, there exist points $y^{\nu 1} < y^{\nu 2} < y^{\nu 3}$ such that $v(y^{\nu 1}) < -1 + \theta\alpha, v(y^{\nu 2}) > 1 - \theta\alpha$, and $v(y^{\nu 3}) < 1 - \alpha$. Hence using the fact that Q is nondecreasing (as the antiderivative of the positive function $\sqrt{2F}$),

$$\begin{aligned}
\kappa_1(1+\kappa_4) &\overset{(3.9),(3.11)}{\ge} \int_{B_\nu(\rho)} |\partial_{y^\nu}(Q \circ v)| \\
&\ge |\int_{y^{\nu 1}}^{y^{\nu 2}} \partial_{y^\nu}(Q \circ v) dy^\nu| + |\int_{y^{\nu 2}}^{y^{\nu 3}} \partial_{y^\nu}(Q \circ v) dy^\nu| \\
&\ge |Q(1-\theta\alpha) - Q(-1+\theta\alpha)| + |Q(1-\alpha) - Q(1-\theta\alpha)| \\
&\ge |Q(1-\theta\alpha) - Q(-1+\theta\alpha)| + 2\kappa_1\kappa_4
\end{aligned}$$

if we choose $\kappa_4 := \frac{1}{2\kappa_1}|Q(1-\alpha) - Q(1-\frac{\alpha}{2})|$, since we have said that $\theta \le \frac{1}{2}$. This inequality is false when $\theta = 0$, since $\kappa_1 = Q(1) - Q(-1)$, and so it also fails for sufficiently small $\theta \in (0, \frac{1}{2})$. Hence we can choose κ_3 small enough to obtain a contradiction.

4. We now replace v on the interval $(\frac{3\rho}{4}, \rho)$ by the minimizer of the functional

$$w \mapsto \int_{\frac{3\rho}{4}}^{\rho} e_{\varepsilon,\nu}(w)\ dy^\nu$$

subject to the boundary conditions $w(\frac{3\rho}{4}) = v(\frac{3\rho}{4})$ and $w(\rho) = v(\rho)$. Let v_1 denote the function obtained in this way. Standard maximum principle arguments[5] imply that $v_1(\frac{7}{8}\rho) \ge 1 - Ce^{-c/\varepsilon}$. In a similar way, we can modify v_1 on $(-\rho, -\frac{3\rho}{4})$ to obtain a function v_2 with less energy than that of v_1, and such that $v_2(-\rho) = v(-\rho)$, and $v_2(-\frac{7}{8}\rho) \le -1 + Ce^{-c/\varepsilon}$.

[5]The point is that one can easily check that $w(y^\nu) := 1 - \alpha\frac{\cosh(b(y^\nu - (7\rho/8))/\varepsilon)}{\cosh(b\rho/8\varepsilon)}$ satisfies $-w'' + \varepsilon^{-2} f(w) \le 0$ in $(\frac{3\rho}{4}, \rho)$, if b is fixed small enough (depending on F). Then in view of (3.12) and the fact that f is decreasing on $(1-\alpha, 1)$, one can use the the maximum principle to find that $v_1 > w$ in $(\frac{3\rho}{4}, \rho)$.

Thus $|Q(v_2(\frac{7}{8}\rho))-Q(1)|\le Ce^{-c/\varepsilon}$, and similarly $|Q(v_2(-\frac{7}{8}\rho))-Q(-1)|\le Ce^{-c/\varepsilon}$. As a result, using (3.9) and (3.11) as in Step 3 and recalling that $\kappa_1=Q(1)-Q(-1)$, we obtain

$$\begin{aligned}\kappa_1(1+\zeta_0)&\overset{(3.9)}{\ge}\int_{B_\nu(\rho)}\varepsilon\, e_{\varepsilon,\nu}(v)dy^\nu\ge\int_{B_\nu(\rho)}\varepsilon\, e_{\varepsilon,\nu}(v_2)dy^\nu\\&\ge|Q(v_2(-\rho))-Q(v_2(-\frac{7}{8}\rho))|+|Q(v_2(-\frac{7}{8}\rho))-Q(v_2(\frac{7}{8}\rho))|+|Q(v_2(\frac{7}{8}\rho))-Q(v_2(\rho)|\\&\ge|Q(v_2(-\rho))-Q(-1)|+\kappa_1+|Q(1))-Q(v_2(\rho)|-Ce^{c/\varepsilon}.\end{aligned}$$

This implies (3.8). Also, since $v_2=v$ at $\pm\rho$, the above implies that

$$\begin{aligned}Q(v(\rho))-Q(v(-\rho))&=\kappa_1+Q(v(\rho))-Q(1)-[Q(v(-\rho))-Q(-1)]\\&\ge\kappa_1-|Q(v(\rho))-Q(1)|-[Q(v(-\rho))-Q(-1)]\\&\ge\kappa_1(1-\zeta_0)-Ce^{-c/\varepsilon}.\end{aligned}\tag{3.13}$$

5. We now use (3.13) to prove (3.10). First note that

$$\begin{aligned}\int_{B_\nu(\rho)}\left|\frac{\varepsilon}{2}v_{y^\nu}^2-\frac{1}{\varepsilon}F(v)\right|dy^\nu&\le\int_{B_\nu(\rho)}\left|\sqrt{\varepsilon}v_{y^\nu}-\frac{\sqrt{2F(v)}}{\sqrt{\varepsilon}}\right|\ \left|\sqrt{\varepsilon}v_{y^\nu}+\frac{\sqrt{2F(v)}}{\sqrt{\varepsilon}}\right|\ dy^\nu\\&\le C\left(\int_{B_\nu(\rho)}\left|\sqrt{\varepsilon}v_{y^\nu}-\frac{\sqrt{2F(v)}}{\sqrt{\varepsilon}}\right|^2\ dy^\nu\right)^{1/2}\left(\int_{B_\nu(\rho)}\varepsilon\ e_{\varepsilon,\nu}(v)\ dy^\nu\ \right)^{1/2}\end{aligned}$$

Expanding the square and recalling that $\sqrt{2F}=Q'$, we see that

$$\begin{aligned}\int_{B_\nu(\rho)}\frac{1}{2}\left|\sqrt{\varepsilon}v_{y^\nu}-\frac{\sqrt{2F(v)}}{\sqrt{\varepsilon}}\right|^2\ dy^\nu&=\int_{B_\nu(\rho)}e_{\varepsilon,\nu}(v)dy^\nu-\int_{B_\nu(\rho)}Q'(v)v_{y^\nu}\ dy^\nu\\&=\int_{B_\nu(\rho)}e_{\varepsilon,\nu}(v)dy^\nu-[Q(v(\rho))-Q(v(-\rho))]\\&\overset{(3.9),(3.13)}{\le}\kappa_1(1+\zeta_0)-(\kappa_1(1-\zeta_0)-Ce^{-c/\varepsilon})\\&\le C\zeta_0+Ce^{-c/\varepsilon}.\end{aligned}$$

Combining these inequalities and again appealing to (3.9), we arrive at (3.10). □

The next Lemma is used to establish requirement (1.32) as discussed in the Introduction. In this lemma we write v as a function of two variables, y^0 and y^ν.

Lemma 6. *Let $B_\nu(\rho)\subset\mathbb{R}$ be an interval as above, and let $v\in H^1((0,\tau)\times B_\nu(\rho))$ for some $\tau>0$. Then there exists a constant C, depending on ρ but independent of τ and of $\varepsilon\in(0,1]$, such that*

$$\int_{B_\nu(\rho)}|y^\nu|\ |v(0,y^\nu)-v(\tau,y^\nu)|^2\ dy^\nu\le C\int_{(0,\tau)\times B_\nu(\rho)}\frac{\varepsilon}{2}v_{y^0}^2+\frac{(y^\nu)^2}{\varepsilon}F(v)\ dy^\nu\ dy^0.$$

Proof. **1**. For $Q:\mathbb{R}\to\mathbb{R}$ as above such that $Q'(s)=\sqrt{2F(s)}$,

$$\frac{\varepsilon}{2}v_{y^0}^2+\frac{y^2}{\varepsilon}F(v) \quad\geq\quad |y^\nu|\sqrt{2F(v)}|v_{y^0}| \quad=\quad |y^\nu|\,|Q(v)_{y^0}|.$$

By integrating this inequality, we find that

$$\begin{aligned}\int_{(0,\tau)\times B_\nu(\rho)}\frac{\varepsilon}{2}v_{y^0}^2+\frac{y^2}{\varepsilon}F(v)\;dy^\nu\;dy^0 \quad&\geq\int_{B_\nu(\rho)}\int_0^\tau|y^\nu|\;\left|Q(v)_{y^0}\right|\,dy^0\,dy^\nu\\ &\geq\int_{B_\nu(\rho)}|y^\nu|\;|Q(v(\tau,y^\nu))-Q(v(0,y^\nu))|dy^\nu.\end{aligned}$$

Finally, our assumption (1.9) that $F(s)\geq(1-|s|)^2$ and elementary calculus imply that

$$|Q(b)-Q(a)|\geq c(b-a)^2,$$

and the lemma follows. □

Now we can give the

Proof of Proposition 3. Since the equation is well-posed in $H^1\times L^2$, and since all the quantities in the statement are continuous in $H^1\times L^2$, we may prove the Proposition for v smooth.

In the proof we will write simply $\mathcal{D}(\cdot)$ instead of $\mathcal{D}(\cdot;\rho_1/2)$.

Step 1. We may assume that $s_1=0$. We will use the notation $s_{\max}:=\min\{\rho_1/2c_*,T_1\}$ and

$$W_\nu(s):=B_\nu(\rho_1-c_*s),\qquad\qquad W(s):=\mathbb{T}^n\times W_\nu(s).$$

We define

$$\begin{aligned}\zeta_1(s)&:=\delta_\varepsilon\int_{\{s\}\times W(s)}(1+(y^\nu)^2)\,e_\varepsilon(v)\;dy'-1\\ \zeta_2(s)&:=\mathcal{D}(v(s))\\ \zeta_3(s)&:=\delta_\varepsilon\int_{\{s\}\times W(s)}|D_\tau v|^2+(y^\nu)^2\left[|\nabla_\nu v|^2+\frac{1}{\varepsilon^2}F(v)\right]\;dy'.\end{aligned}$$

We first claim that

$$\zeta_1(s)\leq C\zeta_0+C\int_0^s\zeta_3(\sigma)d\sigma\qquad\qquad\text{for } s\in(0,s_{max}]. \tag{3.14}$$

Towards this end we compute

$$\begin{aligned}\zeta_1'(s)=I_1-c_*I_2,&\qquad\text{where}\\ I_1&:=\delta_\varepsilon\int_{\{s\}\times W(s)}(1+(y^\nu)^2)\,\frac{\partial}{\partial y^0}e_\varepsilon(v)\;dy'\\ I_2&=\delta_\varepsilon\int_{\{s\}\times\mathbb{T}^n\times\partial W_\nu(s)}(1+(y^\nu)^2)\,e_\varepsilon(v)\;dy^{\tau\prime}.\end{aligned}$$

To estimate I_1, we use Lemma 2 and integrate by parts in the spatial variables. From (2.28) we easily see that $|y^\nu|\,|\varphi^\nu| \le C(|D_\tau v|^2 + (y^\nu)^2|\nabla_\nu v|^2)$. Thus we arrive at

$$I_1 \le C\delta_\varepsilon \int_{\{s\}\times W(s)} (|D_\tau v|^2 + (y^\nu)^2|\nabla_\nu v|^2)\ dy' \\ + \delta_\varepsilon \int_{\{s\}\times\mathbb{T}^n\times\partial W_\nu(s)} (1+(y^\nu)^2)\ |\varphi^N|\ dy^{\tau\prime}.$$

Our choice (3.4) of c_* exactly guarantees that $|\varphi_N| \le c_* e_\varepsilon(v)$, so that the boundary term above is dominated by $-c_* I_2$. It follows that $\zeta_1' \le C\zeta_3$. Since it is clear from (3.5) that $\zeta_1(0) \le \zeta_0$, we conclude that (3.14) holds.

Step 2. Next, we estimate ζ_2. Using the hypotheses and Lemma 6 we find that

$$\begin{aligned}\zeta_2(s) &\le 2\mathcal{D}(v(0)) + 2\int_{\mathbb{T}^n\times B_\nu(\rho_1/2)} |y^\nu|\ |v(s,y') - v(0,y')|^2\ \ dy' \\ &\le 2\zeta_0 + C\int_{\mathbb{T}^n}\left(\int_0^s \int_{\mathbb{T}^n\times B_\nu(\rho_1/2)} \frac{\varepsilon}{2}|v_{y^0}|^2 + \frac{(y^\nu)^2}{\varepsilon}F(v)\ dy^\nu\ dy^0\right) dy^{\tau\prime} \\ &\le 2\zeta_0 + C\int_0^s \zeta_3(\sigma)d\sigma \qquad (3.15)\end{aligned}$$

for $s \le s_{max}$. We have changed the order of integration and used the fact that $\mathbb{T}^n \times B_\nu(\rho_1/2) = W_\nu(\rho_1/2c_*) \subset W_\nu(s)$ for $s \le s_{max} \le \rho_1/2c_*$.

Step 3. Finally, we claim that

$$\zeta_3(s) \le C\left(\zeta_1(s)\ +\ \zeta_2(s)\ +\ e^{-C/\varepsilon}\right) \qquad (3.16)$$

for every $s \in (0, s_{max}]$. We fix such an s, and we often write $v(\cdot)$ instead of $v(s,\cdot)$. Note that (2.25) implies that

$$(1+(y^\nu)^2)e_\varepsilon(v) \ge \frac{1}{2}\lambda|D_\tau v|^2 + (1+(y^\nu)^2)e_{\varepsilon,\nu}(v).$$

It follows from this and the definitions of ζ_1, ζ_3 that

$$\zeta_1(s) \ge c\ \zeta_3(s) + \quad \delta_\varepsilon\int_{\{s\}\times W(s)} e_{\varepsilon,\nu}(v)\ dy'\ -\ 1.$$

Thus it suffices to show that

$$1 - \delta_\varepsilon\int_{\{s\}\times W(s)} e_{\varepsilon,\nu}(v)\ dy' \quad\le\quad C\zeta_2(s)\ +\ Ce^{-c/\varepsilon}. \qquad (3.17)$$

To do this, let us say that a point $y^{\tau\prime} \in \mathbb{T}^n$ is *good* if

$$\mathcal{D}_\nu(v(y^{\tau\prime})) \le \kappa_3$$

and bad otherwise, for $v(y^{\tau\prime})(y^\nu) := v(y^{\tau\prime}, y^\nu)$. Then Chebyshev's inequality implies that

$$|\{y^{\tau\prime} \in \mathbb{T}^n\ : y^{\tau\prime} \text{ is bad }\}| \le \frac{1}{\kappa_3}\int_{\{s\}\times\mathbb{T}^n} \mathcal{D}_\nu(v(y^{\tau\prime}))dy^{\tau\prime} = C\mathcal{D}(v(s)) = C\zeta_2(s). \qquad (3.18)$$

Thus $|\{y^{\tau\prime} \in \mathbb{T}^n \ : y^{\tau\prime} \text{ is good}\}| \geq 1 - C\zeta_2(s)$, and so Lemma 5 implies that

$$\begin{aligned}
&\delta_\varepsilon \int_{\{s\}\times W(s)} e_{\varepsilon,\nu}(v)\ dy' \\
&\qquad \geq \int_{\{(s,y^{\tau\prime}):y^{\tau\prime}\in\mathbb{T}^n\text{is good}\}} \left(\delta_\varepsilon \int_{W_\nu(s)} e_{\varepsilon,\nu}(v)\ dy^\nu \right) dy^{\tau\prime} \\
&\qquad \overset{(3.8)}{\geq} (1 - C\zeta_2(s))\,(1 - Ce^{-c/\varepsilon}). \qquad (3.19)
\end{aligned}$$

This proves (3.17), and hence (3.16).

Step 4. By combining the previous few steps and recalling that $\zeta_0 \geq \varepsilon^2$, we see that

$$\zeta_3(s) \leq C\zeta_0 + C\int_0^s \zeta_3(\sigma)d\sigma,$$

so Gronwall's inequality implies that that there exists some C such that $\zeta_3(s) \leq C\zeta_0$ for all $s \in (0, \rho_1/2c_*)$. Then (3.14) and (3.15) imply that $\zeta_1(s), \zeta_2(s) \leq C\zeta_0$. These estimates imply all the conclusions of the proposition. □

4. INITIAL ENERGY ESTIMATES, $k = 1$.

In this section we indicate how to modify the above arguments to obtain control over v on a portion of a hypersurface of the form $\{y^0 = \text{constant}\}$, starting from our assumptions (2.31)–(2.36) about u at $t = 0$, which translate to information about v on a hypersurface of the form $\{(b(y'), y') : y' \in \mathbb{T}^n \times B_\nu(\rho_0)\}$, with b in general a non-constant function. (Recall that the function b was found in Lemma 3). This is in general needed before we can start to iterate Proposition 3.

We note that if we assume that the minimal surface Γ has velocity 0 at time $t = 0$, then it is easy to check that $b(y') \equiv 0$. As a result, the hypotheses (3.5), (3.6) of Proposition 3 follow immediately in this case from our general assumptions (2.31), (2.34)-(2.36) on the initial data. So the reader who is wiling to accept this restriction on Γ can skip this section (and Section 5.3) without any loss.

We continue to follow the notational conventions for the case $k = 1$, summarized at the beginning of Section 3. We will prove

Proposition 4. *Assume that $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}$ is a solution of (2.26) on with data that satisfies (2.34)–(2.36) on the hypersurface $\{(b(y'), y') : y' \in \mathbb{T}^n \times B_\nu(\rho_0)\}$.*

Then there exists some $s_1 > 0$ and $\rho_1 > 0$ for which v satisfies the hypotheses (3.5), (3.6) of Proposition 3, with ζ_0 replaced by $C\zeta_0$, and such that in addition

$$\delta_\varepsilon \int_{\{y\in(-T_1,s_1)\times\mathbb{T}^n\times B_\nu(\rho_1):\psi^0(y)>0\}} \left[|D_\tau v|^2 + |y^\nu|^2(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2}F(v)) \right]\ dy \leq C\zeta_0.$$

If we simply tried to repeat our earlier arguments, we would have to worry about the way in which a cone with slope c_* intersects the initial hypersurface, and these considerations would force us to impose unnatural restrictions on the initial velocity of the surface Γ. We therefore exploit finite propagation speed in a different and sharper way than in our earlier

arguments. (We could have done this earlier, but we wanted to present our basic estimate in a relatively simple setting.) This and other considerations force us to introduce a certain amount of notation.

We start by defining

$$\mathcal{C} := \{(t,x) \in \mathbb{R}^{1+N} : \mathrm{dist}(x,\Gamma_0) < \tau - t, t > 0\}. \tag{4.1}$$

where dist denotes the Euclidean distance function, $\Gamma_0 = \{H(0, y^{\tau\prime}) : y^{\tau\prime} \in \mathbb{T}^n\}$, and $\tau > 0$ is chosen so small that

$$\mathcal{C} \subset\subset \mathrm{Image}(\psi). \tag{4.2}$$

Note that $\mathcal{C}$ consists of the set of points for which the solution of the semilinear wave equation (1.1) depends solely on the data in the set $\mathcal{C}_0 := \{x \in \mathbb{R}^N : \mathrm{dist}(x,\Gamma_0) < \tau\}$. We continue by defining

$$V := \psi^{-1}(\mathcal{C}),$$
$$s_0 := \inf\{y^0 \in (-T_1, T_1) \ : \ [\{y^0\} \times \mathbb{T}^n \times B_\nu(\rho_0)] \cap V \neq \emptyset\}$$

and

$$V^* := \{y = (y^0, y') \in (s_0, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \ : \ (s, y') \in V \text{ for some } s \geq y^0\}$$

Thus V^* is just V "extended downward" in the timelike y^0 variable to s_0. For $s \in R$ we define

$$V(s) := \{y \in V : y^0 < s\}, \qquad V^*(s) := \{y \in V^* : y^0 < s\}.$$

We further define

$$\begin{aligned}
\partial_0 V(s) &:= \{y \in \partial V(s) \ : \psi^0(y) = 0\},\\
\partial_1 V(s) &:= \{y = (y^0, y') \in \partial V(s) \ : y^0 = s\},\\
\partial_2 V(s) &:= \partial V(s) \setminus (\partial_0 V(s) \cup \partial_1 V(s).)
\end{aligned}$$

We will also write

$$\begin{aligned}
\partial_1 V^*(s) &:= \{y = (y^0, y') \in \partial V^*(s) \ : y^0 = s\}\\
\partial_0 V &:= \{y \in \partial V \ : \psi^0(y) = 0\}\\
W_0 &:= \{y' \in \mathbb{T}^n \times B_\nu(\rho_0) : (y^0, y') \in \partial_0 V \text{ for some } y^0\}.
\end{aligned}$$

Finally we define

$$W_i(s) := \{y' \in \mathbb{T}^n \times B_\nu(\rho_0) : (y^0, y') \in \partial_i V(s) \text{ for some } y^0\}$$

for $i = 0, 1, 2$, and similarly $W_i^*(s)$.

The next lemma collects some geometric facts that we will need about the sets defined above.

Lemma 7.

$$(W_0(s) \setminus W_1(s)) \cap W_1^*(s) = \emptyset \quad \textit{for all } s. \tag{4.3}$$

In addition, there exists $s_1 > 0$ and $\rho_1 > 0$ such that

$$(s_0, s_1) \times \mathbb{T}^n \times B_\nu(\rho_1) \subset V^* \qquad \textit{and} \quad \{s_1\} \times \mathbb{T}^n \times B_\nu(\rho_1) \subset V. \tag{4.4}$$

proof of Lemma 7. To prove (4.3), fix $y' \in W_0(s) \setminus W_1(s)$. The definitions imply that the line $\{(y^0, y') : y^0 \in \mathbb{R}\}$ intersects $\partial_0 V(s)$ and does not meet $\partial_1 V(s)$, so it must leave $\bar{V}$ at a point (σ, y') with $\sigma < s$. Arguments like those of Lemma 3 show that once the line has left $\bar{V}$, it cannot re-enter, as if it did, the timelike curve $s \mapsto X(s) := \psi(s, y')$ (see Lemma 3) would intersect $\partial^+\mathcal{C} := \{(t,x) \in \partial\mathcal{C} : t > 0\}$ more than once, which is impossible. Thus the line does not intersect $\bar{V}^*$ at any point (y^0, y') with $y' > \sigma$, and so it cannot intersect $\partial_1 V^*(s) \subset \{((y^0, s) \in \bar{V}^* : y^0 = s\}$. Thus $y' \notin W_1^*(s)$, proving (4.3).

Next, the existence of $s_1, \rho_1 > 0$ satisfying (4.4) follows from the fact that the (Euclidean) distance from $\{0\} \times \mathbb{T}^n \times \{0\} = \psi^{-1}(\Gamma_0)$ to $\partial^+ V := \partial V \setminus \partial_0 V = \psi^{-1}(\partial^+\mathcal{C})$ is positive. This last fact in turn is clear from the fact that the distance from Γ_0 to $\partial^+\mathcal{C}$ is positive, together with the smoothness of ψ. □

Recall that $v_0 : \mathbb{T}^n \times B_\nu(\rho_0) \cong \{0\} \times \mathbb{T}^n \times B_\nu(\rho_0)$ was defined in (2.33). We extend v_0 to $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho)$ such that it is independent of y^0; this extended function is still denoted v_0.

The remainder of this section contains the proof of Proposition 4. In the proof, when we want to distinguish between row vectors and column vectors (which one can think as vectors and covectors, respectively), we will write $\bar{\xi}$ to denote a column vector, with components ξ^α, and $\underline{\xi}$ for a row vector, with components ξ_α.

Proof of Proposition 4. As in Proposition 3, it suffices to prove the proposition for v smooth in $\bar{V}$.

Step 1. We define $v^* : V^* \to \mathbb{R}$ by

$$v^*(y) = \begin{cases} v(y) & \text{if } y \in V \\ v_0(y) & \text{if } y \in V^* \setminus V. \end{cases} \tag{4.5}$$

Since $v = v_0$ on $\bar{V} \cap (V^* \setminus V) = \partial_0 V$, it is easy to see that v^* is Lipschitz in V^*. Note however that the derivatives of v^* are in general discontinuous across $\partial_0 V$.

We define

$$\begin{aligned}
\zeta_1(s) &:= \delta_\varepsilon \int_{\partial_1 V^*(s)} (1 + (y^\nu)^2)\, e_\varepsilon(v^*)\ dy' \ - 1 \\
\zeta_2(s) &:= \mathcal{D}(v^*(s); \rho_1/2) \\
\zeta_3(s) &:= \delta_\varepsilon \int_{\partial_1 V^*(s)} \left[|D_\tau v^*|^2 + (y^\nu)^2\, e_{\varepsilon,\nu}(v^*) \right]\ dy'.
\end{aligned}$$

In view of (4.4), we can repeat word for word the arguments from the proof of Proposition 3 to find that

$$\zeta_3(s) \le C \left(\zeta_1(s) + \zeta_2(s) + e^{-c/\varepsilon} \right) \tag{4.6}$$

and

$$\zeta_2(s) \le 2\zeta_2(s_0) + C \int_{s_0}^s \zeta_3(\sigma)\ d\sigma$$

for every $s \in [s_0, s_1]$. And the definition of s_0 implies that $v^* = v_0$ on $\partial_1 V^*(s_0) := \{s_0\} \times W_0$, so that $\zeta_2(s_0) \le \zeta_0$ by (2.36). Thus

$$\zeta_2(s) \le C\zeta_0 + C\int_{s_0}^{s} \zeta_3(\sigma)\, d\sigma \tag{4.7}$$

for every $s \in [s_0, s_1]$.

The remainder of the proof is devoted to the estimate of ζ_1. Since v^* is smooth away from $\partial_0 V$ and (by Fubini's Theorem) $\partial_1 V^*(s) \cap \partial_0 V$ has $\mathcal{H}^N$ measure 0 for $\mathcal{L}^1$ a.e. s, the definition of v^* implies that

$$e_\varepsilon(v^*) = \begin{cases} e_\varepsilon(v) & \mathcal{H}^N \text{ a.e. in } \partial_1 V(s) \\ e_\varepsilon(v_0) & \mathcal{H}^N \text{ a.e. } \partial_1 V^*(s) \setminus \partial_1 V(s) \end{cases} \tag{4.8}$$

for a.e. s. Also, if $[\cdots]$ denotes an integrand that does not depend on the y^0 variable, then clearly $\int_{\partial_1^* V(s)\setminus \partial_1 V(s)} [\cdots] dy' = \int_{W_1^*(s)\setminus W(s)} [\cdots] dy'$. Thus

$$\int_{\partial_1 V^*(s)} (1+(y^\nu)^2)\, e_\varepsilon(v^*)\, dy' = \int_{\partial_1 V(s)} (1+(y^\nu)^2)\, e_\varepsilon(v)\, dy' + \int_{W_1^*(s)\setminus W_1(s)} (1+(y^\nu)^2) e_\varepsilon(v_0)\, dy' \tag{4.9}$$

for a.e. s.

Step 2. We claim that for a.e. s,

$$\begin{aligned} \delta_\varepsilon \int_{\partial_1 V(s)} (1+(y^\nu)^2)\, e_\varepsilon(v)\, dy' \le \delta_\varepsilon \int_{\partial_0 V(s)} (1+(y^\nu)^2)\, (-n_0 e_\varepsilon(v) + n_i \varphi^i)\, \mathcal{H}^N(dy) \\ + C\int_{s_0}^{s} \zeta_3(\sigma) d\sigma, \end{aligned} \tag{4.10}$$

where $\underline{n}(y)$ denotes the (Euclidean) outer unit normal at a point $y \in \partial V(s)$, thought of as a row vector with components n_α, and φ^i is defined in (2.28) and appears in the local energy estimate of Lemma 2.

Step 2.1 To prove (4.10) we will first integrate by parts and show that some of the boundary terms have a sign and hence can be discarded. (In this we basically follow the proof of Proposition 3.) For this, it is useful to define $\tilde{\mathcal{T}}_\varepsilon = \tilde{\mathcal{T}}_\varepsilon(v)$ by

$$\tilde{\mathcal{T}}^\alpha_{\varepsilon,\beta} := \delta^\alpha_\beta \left(\frac{1}{2} g^{\gamma\delta} v_{y^\gamma} v_{y^\delta} + \frac{1}{\varepsilon^2} F(v) \right) - g^{\alpha\gamma} v_{y^\gamma} v_{y^\beta}. \tag{4.11}$$

Observe from the definitions that[6]

$$\tilde{\mathcal{T}}^0_{\varepsilon,0}(v) = e_\varepsilon(v) \qquad \text{and} \quad \tilde{\mathcal{T}}^i_{\varepsilon,0}(v) = -\varphi^i \tag{4.12}$$

so that the conclusion of Lemma 2 can be written $\partial_{y^\alpha} \tilde{\mathcal{T}}^\alpha_{\varepsilon,0} \le C(|D_\tau v|^2 + (y^\nu)^2 |\nabla_\nu v|^2)$.

[6]In fact $\tilde{T}_\varepsilon$ is just the energy-momentum tensor for u expressed in terms of the y-cordinates. The fact that the energy-momentum tensor is divergence-free, when written in the y coordinates, takes the form $\partial_{y^\alpha}(\tilde{\mathcal{T}}^\alpha_{\varepsilon,\beta}(v)\sqrt{-g}) = 0 \forall \beta$. One can use this fact to give a proof of Lemma 2 slightly different from the one presented earlier.

We now compute

$$\begin{aligned}
\delta_\varepsilon \int_{V(s)} \partial_{y^\alpha} \left[(1+(y^\nu)^2)\tilde{\mathcal{T}}^\alpha_{\varepsilon,0}\right] \, dy &\le C\delta_\varepsilon \int_{V(s)} \left[(|D_\tau v|^2 + (y^\nu)^2 |\nabla_\nu v|^2) + y^N \tilde{\mathcal{T}}^N_{\varepsilon,0}\right] \, dy \\
&\le C\delta_\varepsilon \int_{V(s)} (|D_\tau v|^2 + (y^\nu)^2 |\nabla_\nu v|^2) \, dy \\
&\le C \int_{s_0}^s \zeta_3(\sigma) d\sigma. \qquad (4.13)
\end{aligned}$$

On the other hand, we can integrate by parts to rewrite the left-hand side as an integral over $\partial V(s)$. Then noting that $\underline{n}(y) = (1, 0, \ldots, 0)$ for $y \in \partial_1 V(s)$, we find that

$$\begin{aligned}
\delta_\varepsilon \int_{V(s)} \partial_{y^\alpha} \left[(1+(y^\nu)^2)\tilde{\mathcal{T}}^\alpha_{\varepsilon,0}\right] \;=\; & \delta_\varepsilon \int_{\partial_1 V(s)} (1+(y^\nu)^2) e_\varepsilon(v) dy' \\
&+ \delta_\varepsilon \int_{\partial_0 V(s)} \left[(1+(y^\nu)^2) n_\alpha \tilde{\mathcal{T}}^\alpha_{\varepsilon,0}\right] \, d\mathcal{H}^N(y) \\
&+ \delta_\varepsilon \int_{\partial_2 V(s)} \left[(1+(y^\nu)^2) n_\alpha \tilde{\mathcal{T}}^\alpha_{\varepsilon,0}\right] \, d\mathcal{H}^N(y).
\end{aligned}$$

By combining this with (4.13) and recalling (4.12), we see that our claim (4.10) will follow if we can show that the last integral on the right-hand side is positive.

Step 2.2. To do this we will show that

$$n_\alpha(y) \; \tilde{\mathcal{T}}^\alpha_{\varepsilon,0}(y) \;\ge\; 0 \qquad \text{for a. e. } y \in \partial_2 V(s). \qquad (4.14)$$

To do this, we first check that

$$g^{\alpha\beta} n_\alpha n_\beta = 0 \qquad \text{a.e. } y \in \partial_2 V. \qquad (4.15)$$

In fact, we will show that this holds at every $y \in \partial_2 V$ such that ∂C has a tangent plane at $x = \psi(y)$; this is a set of full measure. Fix such a y and let $\bar{w} = (w^\alpha)$ be any (column) vector tangent to $\partial\mathcal{C}$ at x. Also, let $\underline{m}(x)$ denote the (Euclidean) outer unit normal to $\mathcal{C}$ at $x \in \partial\mathcal{C}$, again thought of as a row vector with components m_α. Writing $\phi = \psi^{-1}$ as usual, since ϕ maps $\partial\mathcal{C}$ to ∂V, it is clear that $D\phi(x)\,\bar{w}$ is tangent to ∂V at $\phi(x) = y$, which implies that $\underline{n}(y) D\phi(x)\,\bar{w} = 0$. Since this holds for all tangent vectors $\bar{w}$ at x, it follows that $\underline{n}(y)\; D\phi(x)$ is parallel to the Euclidean unit normal $\underline{m}$ to ∂C at x; that is $\underline{n}(y) D\phi(x) = \lambda \underline{m}(x)$ for some $\lambda \in \mathbb{R}$. And the form of $\mathcal{C}$ implies that $\underline{m}$ is a null vector, so that

$$0 = \lambda^2 \eta^{\alpha\beta} m_\alpha m_\beta \;=\; \lambda^2 \eta^{\alpha\beta} n_\gamma \phi^\gamma_\alpha m_\delta \phi^\delta_\beta = g^{\gamma\delta} n_\gamma n_\delta,$$

proving (4.15). Note also that $n_0(y) > 0$ for $y \in \partial_2 V$, and recall further that $F(u) \ge 0$. Thus

$$\begin{aligned}
n_\alpha \tilde{\mathcal{T}}^\alpha_{\varepsilon,0} &= \frac{n_0}{\varepsilon^2} F(u) + \frac{n_0}{2} g^{\alpha\beta} v_{y^\alpha} v_{y^\beta} - v_{y^0} g^{\alpha\beta} n_\alpha v_{y^\beta} \\
&\ge \frac{n_0}{2} g^{\alpha\beta} v_{y^\alpha} v_{y^\beta} - v_{y^0} g^{\alpha\beta} n_\alpha v_{y^\beta} \\
&= \frac{n_0}{2} g^{\alpha\beta} (Dv - \frac{v_{y^0}}{n_0} n)_\alpha (Dv - \frac{v_{y^0}}{n_0} n)_\beta
\end{aligned}$$

using (4.15). If we write $\xi := Dv - \frac{v_{y0}}{n_0} n$, then clearly $\xi_0 = 0$, which implies that

$$g^{\alpha\beta}\xi_\alpha\xi_\beta = g^{ij}\xi_i\xi_j = a^{\alpha\beta}\xi_\alpha\xi_\beta \ge 0.$$

Thus we have proved (4.14).

Step 3. Next we note that

$$-\int_{\partial_0 V(s)} (1+(y^\nu)^2)\, n_0(y)\, e_\varepsilon(v)(y)\mathcal{H}^N(dy) = \int_{W_0(s)} (1+(y^\nu)^2)\, e_\varepsilon(v)(b(y'),y')\; dy, \tag{4.16}$$

where we recall that $\partial_0 V = \{(b(y'),y') \ : \ y' \in W_0\}$, and hence that $\partial_0 V(s) = \{(b(y'),y') \ : \ y' \in W_0(s)\}$. This is obvious, because the Euclidean outer unit normal to $V(s)$ is given by $\underline{n} = (-1,\nabla b)/(1+|\nabla b|^2)^{1/2}$, with the minus sign appearing because V sits above the graph. Thus $-n_0(b(y'),y') = (1+|\nabla b(y')|^2)^{-1/2}$, and then (4.16) follows from a change of variables using the area formula.

Step 4. Now we combine (4.16) with (4.9), (4.10) to find that

$$\zeta_1(s) \le C\int_{s_0}^s \zeta_3(\sigma)\, d\sigma + A + B,$$

for a.e. $s \in [s_0, s_1]$, where

$$A := \delta_\varepsilon \int_{W_0(s)} (1+(y^\nu)^2)(e_\varepsilon(v) - e_\varepsilon(v_0))(b(y'),y')dy' + \delta_\varepsilon \int_{\partial_0 V(s)} (1+(y^\nu)^2)\, n_i\varphi^i \; d\mathcal{H}^N,$$

$$B := \delta_\varepsilon \int_{W_1^*(s)\setminus W_1(s)} (1+(y^\nu)^2)e_\varepsilon(v_0)dy' + \delta_\varepsilon \int_{W_0(s)} (1+(y^\nu)^2)e_\varepsilon(v_0)\; dy' - 1.$$

We have checked in Lemma 7 that $(W_1^*(s)\setminus W_1(s)) \cap W_0(s) = \emptyset$; this is equivalent to (4.3). Thus

$$B \le \delta_\varepsilon \int_{W_0} (1+(y^\nu)^2)e_\varepsilon(v_0)\; dy' - 1 \quad \overset{(2.34)}{\le} \quad \zeta_0.$$

To estimate A, we differentiate the identity $v(b(y'),y') = v_0(y')$ to find that $v_{y^0}\nabla b + \nabla v = \nabla v_0$. Thus $|D(v-v_0)| = |v_{y^0}(1,-\nabla b)| \le C|v_{y^0}|$ at points $(b(y'),y') \in \partial_0 V$, using the control over $\|\nabla b\|_\infty$ obtained in Lemma 3. It follows that at such points

$$e_\varepsilon(v) - e_\varepsilon(v_0) = \frac{1}{2}a^{\alpha\beta}(v-v_0)_{y^\alpha}(v+v_0)_{y^\beta} \le C\left(v_{y^0}^2 + |D_\tau v_0|^2 + |v_{y^0}||\nabla_\nu v_0|\right).$$

Similarly, using (2.19), we see that $|\varphi^i| \le C(v_{y^0}^2 + |D_\tau v_0|^2 + (y^\nu)^2|\nabla_\nu v_0|^2)$, so

$$A \le \quad C\delta_\varepsilon \int_{\partial_0 V} \left(v_{y^0}^2 + |v_{y^0}|\; |\nabla_\nu v_0|\right)\; d\mathcal{H}^N + C\delta_\varepsilon \int_{W_0} (|D_\tau v_0|^2 + (y^\nu)^2|\nabla_\nu v_0|^2)\; dy'.$$

Also, since $v_0(y') = v^*(s_0,y')$,

$$\int_{W_0} \varepsilon(|D_\tau v_0|^2 + (y^\nu)^2|\nabla_\nu v_0|^2)\; dy' \; \le \zeta_3(s_0) \overset{(4.6)}{\le} C(\zeta_1(s_0) + \zeta_2(s_0) + e^{-c/\varepsilon}) \overset{(2.34),(2.35)}{\le} C\zeta_0.$$

Using this fact and (2.35) we conclude that $A \le C\zeta_0$, and hence that

$$\zeta_1(s) \le C\int_{s_0}^s \zeta_3(\sigma)d\sigma + C\zeta_0.$$

Step 5. The rest of the proof exactly follows that of Proposition 3. In the end we find that $\zeta_i(s_1) \le C\zeta_0$ for $i = 1,2,3$, and in view of (4.4), these estimates immediately imply the conclusion of the proposition.

$\square$

5. ENERGY ESTIMATES, $k = 2$

In this section, we prove energy estimates like those from Sections 3, 4 above, but now in the case $k = 2$, so that we consider a vector-valued function $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho) \to \mathbb{R}^2$ solving (2.26), where $B_\nu(\rho) \subset \mathbb{R}^2_\nu$ now denotes a 2-dimensional ball, κ_2 is the constant chosen in (2.23), $\delta_\varepsilon = (\pi|\ln \varepsilon|)^{-1}$, and the nonlinearity in (1.1) is $f = \nabla F$ with $F : \mathbb{R}^2 \to [0,\infty)$ satisfying (1.19).

The main results and proofs in this section are strictly analogous to Propositions 3 and 4. The chief difference is that the "defect confinement functional" $\mathcal{D}$ (discussed in the Introduction) has a quite different form than in the case $k = 1$. Thus, the arguments we need to verify that the desired properties (1.30), (1.32) hold are quite different from, and more delicate than, their counterparts in the scalar case. Once suitable forms of these facts are established, we follow our earlier proofs with only cosmetic changes.

We will use machinery that relates the Jacobian and the Ginzburg-Landau energy. We will give precise statements of the facts from the literature that we need, in the hope of rendering our arguments somewhat accessible to people who are not familiar with these results; see also the book [35] for a general reference on these topics. The results we use (see Lemmas 8, 9, 10) are proved in the sources we cite for $F_{model}(u) = \frac{1}{4}(|u|^2 - 1)^2$, but it is evident[7] from the proofs that they still apply to functions F satisfying the assumptions (1.19) that we impose here.

For $v \in H^1(\mathbb{T}^n \times B_\nu(\rho); \mathbb{R}^2)$ we take $\mathcal{D}$ to have the form (as when $k = 1$)

$$\mathcal{D}(v;\rho) := \int_{\mathbb{T}^n} \mathcal{D}_\nu(v(y^{\tau\prime});\rho)\, dy^{\tau\prime}, \tag{5.1}$$

where $v(y^{\tau\prime})(y^\nu) = v(y^{\tau\prime}, y^\nu)$. And for $w = (w^1, w^2) \in H^1(B_\nu(\rho); \mathbb{R}^2)$, we define

$$\mathcal{D}_\nu(w;\rho) := |||J_\nu w - \pi\delta_0|||_\rho \tag{5.2}$$

where for a measure μ on $B_\nu(\rho)$,

$$|||\mu|||_\rho := \sup\left\{ \int \omega(y^\nu) f(y^\nu) dy^\nu \ : \omega \in C^2_c(B_\rho), |\nabla\omega(y^\nu)| \le |y^\nu|^2, \|\omega\|_{W^{2,\infty}} \le 1 \right\}. \tag{5.3}$$

(Clearly $|||\cdot|||_\rho$ also makes sense for some distributions that are less regular than measures, but we will not need that here.) Here we are using the notation $J_\nu w = \det \nabla_\nu w$. We will also write $\mathbf{J}_\nu w$ for the 2-form $\mathbf{J}_\nu w = J_\nu w\; dy^\nu$, where $dy^\nu := dy^{\nu,1} \wedge dy^{\nu,2}$. Note that

$$\mathbf{J}_\nu w := d_\nu w^1 \wedge d_\nu w^2, \qquad \text{where } d_\nu w^i = \frac{\partial w^i}{\partial y^{\nu,1}} dy^{\nu,1} + \frac{\partial w^i}{\partial y^{\nu,2}} dy^{\nu,2}.$$

[7]In all the proofs we will cite, easy truncation arguments are used to reduce to the case of u such that $|u| \le M$ a.e. for $M = 2$ for example, and then (1.19) implies that $\frac{1}{(C\varepsilon)^2} F_{model}(u) \le \frac{1}{\varepsilon^2} F(u) \le \frac{1}{(\varepsilon/C)^2} F_{model}(u)$. It is then clear that results established for F_{model} carry over to energy functionals that instead contain F, since everything we use is essentially unaffected if ε is replaced by $C\varepsilon$ or ε/C.

(Recall that $y^{\nu,i} = y^{n+i}$.)

General results and heuristics about Jacobians and vortices (see for example [35]), together with the definition of the $|||\cdot|||_\rho$ norm, suggest that if $w : B_\nu(\rho) \to \mathbb{R}^2$ is a function possessing a single "vortex of degree 1" localized near some point in $B_\nu(\rho/2)$, then roughly speaking

$$|||J_\nu w - \pi\delta_0|||_\rho \quad \approx \quad (\text{the distance from the origin to the vortex })^3$$

(The cubic scaling on the right-hand side is related to the condition $|\nabla\omega(y^\nu)| \le |y^\nu|^2$ imposed on test functions in the definition of $|||\cdot|||_\rho$.) Thus, the right-hand side of (5.1) is the average of the above quantity over the tangential y^τ variables.

The first main result of this section parallels Proposition 3 above:

Proposition 5. *Let $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}^2$ satisfy (2.26), where $B_\nu(\rho) \subset \mathbb{R}^2_\nu$ and $f = \nabla F$ with $F : \mathbb{R}^2 \to \mathbb{R}$ satisfying (1.19). Recalling that $\delta_\varepsilon = (\pi \ln \varepsilon|)^{-1}$, assume that there exist $s_1 \in (-T_1, T_1)$, $\rho_1 \in (0, \rho_0)$, and $\zeta_0 \ge \delta_\varepsilon$ such that*

$$\delta_\varepsilon \int_{\{s_1\}\times\mathbb{T}^n\times B_\nu(\rho_1)} (1 + \kappa_2|y^\nu|^2)\, e_\varepsilon(v) dy' - 1 \quad \le \quad \zeta_0 \tag{5.4}$$

$$\mathcal{D}(v(0); \rho_1/2) \quad \le \quad \zeta_0. \tag{5.5}$$

Then there exists a constant C such that

$$\delta_\varepsilon \int_{\{s_1+s\}\times\mathbb{T}^n\times B_\nu(\rho_1 - c_* s)} \left[|D_\tau v|^2 + |y^\nu|^2 (|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2} F(v)) \right] \, dy' \le C\zeta_0$$

$$\delta_\varepsilon \int_{\{s_1+s\}\times\mathbb{T}^n\times B_\nu(\rho_1 - c_* s)} e_\varepsilon(v)(1 + \kappa_2|y^\nu|^2) dy' - 1 \ \le C\zeta_0$$

and

$$\mathcal{D}(v(s); \rho_1/2) \quad \le \quad C\zeta_0$$

for all $s \in [0, \rho_1/2c_*]$ such that $s_1 + s < T_1$. Here c_* is defined in (3.4).

As remarked earlier, there does not exist any initial data satisfying (5.4), (5.5) with $\zeta_0 \ll \delta_\varepsilon$ when $k = 2$, so the condition $\zeta_) \ge \delta_\varepsilon$ is not restrictive.

The second main result of this section parallels Proposition 4.

Proposition 6. *Assume that $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}^2$ is a solution of (2.26) with data that satisfies (2.34)-(2.36) on the hypersurface $\{(b(y'), y') : y' \in \mathbb{T}^n \times B_\nu(\rho_0)\}$, with $\zeta_0 \ge \delta_\varepsilon$ and with $\mathcal{D}$ as defined in (5.1).*

Then there exists some $s_1 > 0$ and $\rho_1 > 0$ for which v satisfies the hypotheses (5.4), (5.5) of Proposition 5, with ζ_0 replaced by $C\zeta_0$, and such that in addition

$$\delta_\varepsilon \int_{\{y\in(-T_1,s_1)\times\mathbb{T}^n\times B_\nu(\rho_1):\psi^0(y)>0\}} \left[|D_\tau v|^2 + |y^\nu|^2 (|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2} F(v)) \right] \, dy \le C\zeta_0.$$

5.1. **variational stability estimates.** We start by establishing some properties relating the $|||\cdot|||_\rho$ norm of the Jacobian Jv and the Ginzburg-Landau energy $e_{\varepsilon,\nu}(v)$. These show will be used to show that $\mathcal{D}(\cdot)$ satisfies requirements (1.30) and (1.32) from the Introduction.

Our first result is analogous to Lemma 5 and establishes a form of (1.30). It is a straightforward consequence of the Jacobian machinery mentioned above.

Proposition 7. *For $\rho > 0$ there exist constant κ_3 and C, both depending on ρ, such that if $w \in H^1(B_\nu(\rho);\mathbb{R}^2)$ and if*

$$\mathcal{D}_\nu(w;\rho) \;=\; |||J_\nu w - \pi\delta_0|||_\rho \;\le\; \kappa_3 \tag{5.6}$$

then

$$|\ln\varepsilon|^{-1}\int_B e_{\varepsilon,\nu}(w)\ dy^\nu \ge \pi - |\ln\varepsilon|^{-1}C. \tag{5.7}$$

The proof of Proposition 7 uses the following facts:

Lemma 8. *Suppose that $\varepsilon \in (0,1]$, that $w \in H^1(B_\nu(\rho);\mathbb{R}^2)$, and that*

$$\|J_\nu w - \pi\delta_0\|_{W^{-1,1}(B_\nu(\rho))} \le \frac{\rho}{10}.$$

Then

$$\frac{1}{|\ln\varepsilon|}\int_{B_\nu(\rho)} e_{\varepsilon,\nu}(w)\ dy^\nu \;\ge\; \pi - \frac{C}{|\ln\varepsilon|}.$$

This follows for example from a much sharper estimate proved in [22], see Theorem 1.3. A slightly different norm is used there in place of the $W^{-1,1}$ norm, but that result is easily seen to imply the one stated here.

Lemma 9. *Suppose that $\varepsilon \in (0,1]$ and that $w \in H^1(B_\nu(\rho);\mathbb{R}^2)$ satisfies*

$$\frac{1}{|\ln\varepsilon|}\int_B e_{\varepsilon,\nu}(w)\ dy^\nu \quad \le 3\pi/2.$$

Then there exists an integer $\ell \in \{0,\pm 1\}$ and a point $\xi \in B$ such that

$$\|J_\nu w - \pi\ell\delta_\xi\|_{W^{-1,1}(B_\nu(\rho))} \le C|\ln\varepsilon|\varepsilon^{1/4}.$$

This follows from Theorem 1.1 in [22]. Using the lemmas we give the

proof of Proposition 7. **1**. Fix $w \in H^1(B_\nu(\rho);\mathbb{R}^2)$. We may assume that

$$\frac{1}{|\ln\varepsilon|}\int_{B_\nu(\rho)} e_{\varepsilon,\nu}(w)\ dy^\nu \;\le\; 3\pi/2, \tag{5.8}$$

since otherwise (5.7) is immediate. So in view of Lemma 8 it suffices to show that there exists a constant $\kappa_3(\rho)$ such that if (5.8) holds and $|||J_\nu w - \pi\delta_0|||_\rho < \kappa_3$, then

$$\|J_\nu w - \pi\delta_0\|_{W^{-1,1}(B_\nu(\rho))} \le \frac{\rho}{10}. \tag{5.9}$$

In fact it suffices to show that there exists some $\varepsilon_0 > 0$ such that the above conclusion holds if $\varepsilon \in (0, \varepsilon_0)$ in (5.8), since we can arrange that (5.7) holds for $\varepsilon > \varepsilon_0$ by choosing C large enough.

Now (5.8) and Lemma 9 imply that there exist an integer ℓ with $|\ell| \leq 1$ and a point $\xi \in B_\nu(\rho)$ such that $\|J_\nu w - \pi\ell\delta_\xi\|_{W^{-1,1}(B_\nu(\rho))} \leq C|\ln\varepsilon|\varepsilon^{1/4}$. Fix a function $\omega_* \in C^2_c(B)$, with $|\nabla\omega_*(y)| \leq |y|^2$ and $\|\omega_*\|_{W^{2,\infty}} \leq 1$, and such that $\omega_*(y) < \omega_*(0)$ if $y \neq 0$. Then (5.6) and the definition of the $|||\cdot|||_\rho$ norm imply that

$$\int \omega_*\ J_\nu w\ dy^\nu - \pi\omega_*(0) \geq -\kappa_3.$$

On the other hand, the estimate $\|J_\nu w - \pi\ell\delta_\xi\|_{W^{-1,1}(B)} \leq C|\ln\varepsilon|\varepsilon^{1/4}$ implies that

$$\int \omega_*\ J_\nu w\ dy^\nu - \pi\ell\omega_*(\xi) \leq C\|\omega_*\|_{W^{1,\infty}}|\ln\varepsilon|\varepsilon^{1/4} \leq C|\ln\varepsilon|\varepsilon^{1/4}.$$

Thus

$$\ell\omega_*(\xi) \geq \omega_*(0) - \frac{\kappa_3}{\pi} - C|\ln\varepsilon|\varepsilon^{1/4}. \tag{5.10}$$

Since $\omega_*(0) > 0$, this implies that $\ell = 1$ for all sufficiently small $\varepsilon > 0$, if κ_3 is fixed small enough. Then $\|Jw(\tau) - \pi\delta_\xi\|_{W^{-1,1}(B_\nu(\rho))} \leq C|\ln\varepsilon|\varepsilon^{1/4}$, and as a result,

$$\begin{aligned}\|J(w(\tau)) - \pi\delta_0\|_{W^{-1,1}(B_\nu(\rho))} &\leq C|\ln\varepsilon|\varepsilon^{1/4} + \pi\|\delta_\xi - \delta_0\|_{W^{-1,1}(B_\nu(\rho))}\\ &\leq C|\ln\varepsilon|\varepsilon^{1/4} + \pi|\xi|\end{aligned}$$

where the last inequality follows immediately from the definition of the $W^{-1,1}$ norm. Since ω_* is continuous and achieves its maximum exactly at the origin, (5.10) implies that if we fix κ_3 still smaller if necessary, then $\pi|\xi| < \rho/20$, and as a result (5.9) holds, for all small ε. □

The second result about the $|||\cdot|||_\rho$ norm is analogous to Lemma 5 and establishes a form of requirement (1.32); in fact, the norm is designed exactly so that an estimate of the form (5.11) holds. In the lemma we write v as a function of $(y^0, y^\nu) \in \mathbb{R} \times \mathbb{R}^2_\nu$

Proposition 8. *Let $v \in H^1((0,\tau) \times B_\nu(\rho); \mathbb{R}^2)$ for some $\rho, \tau > 0$. Then there exist positive constants C, α, depending on ρ but independent of τ and of $\varepsilon \in (0,1]$, such that*

$$\begin{aligned}|||J_\nu v(\tau,\cdot) - J_\nu v(0,\cdot)|||_\rho \leq C\delta_\varepsilon &\int_{(0,\tau)\times B_\nu(\rho)} (|y^\nu|^2 + \varepsilon^\alpha)(|\frac{|Dv|^2}{2} + \frac{1}{\varepsilon^2}F(v))\ dy^\nu\, dy^0\\ &+ C\varepsilon^\alpha\left(1 + \int_{\{0\}\times B_\nu(\rho)} e_{\varepsilon,\nu}(v)dy^\nu + \int_{\{\tau\}\times B_\nu(\rho)} e_{\varepsilon,\nu}(v)dy^\nu\right).\end{aligned} \tag{5.11}$$

We believe that the ε^α in the first integral on the right-hand side of (5.11) could be removed with some work, but the estimate is false without the boundary terms in the second line of (5.11). In any case, all these terms will be negligible in our later arguments.

The proof of Proposition 8 requires the following Lemma.

Lemma 10. *There exist universal constants $C, \alpha > 0$ such that, given any $U \subset \mathbb{R}^3 = \mathbb{R}_{y^0} \times \mathbb{R}^2_\nu$, and $w \in H^1(U;\mathbb{R}^2)$,*

$$\left|\int_U \omega \wedge \mathbf{J}w\right| \leq \frac{C}{|\ln\varepsilon|}\int_U |\omega|\left(\frac{|Dw|^2}{2} + \frac{F(w)}{\varepsilon^2}\right)$$

$$+ C\varepsilon^\alpha(1 + \|D\omega\|_\infty)\left(1 + \|\omega\|_\infty + \int_U (|\omega|+1)\left(\frac{|Dw|^2}{2} + \frac{F(w)}{\varepsilon^2}\right)\right) \tag{5.12}$$

for every compactly supported Lipschitz continuous 1-form ω in U and every $\varepsilon \in (0,1]$. Here $\mathbf{J}w$ denotes the 2-form $dw^1 \wedge dw^2 = (w^1_{y^0}dy^0 + d_\nu w^1) \wedge (w^2_{y^0}dy^0 + d_\nu w^2)$.

This is Lemma 9 of [18], with notation adapted to our setting. In (5.12), Dw denotes as usual the gradient in all 3 variables.

Proof of Proposition 8. **1**. Fix $v \in H^1((0,\tau)\times B_\nu(\rho);\mathbb{R}^2)$. In order to prove (5.11), we must estimate

$$\int_{B_\nu(\rho)} \omega[J_\nu v(\tau,y^\nu) - J_\nu v(0,y^\nu)]\; dy^\nu$$

for an arbitrary $\omega \in C^\infty_c(B_\nu(\rho))$ such that $|\nabla\omega(y)| \leq |y|^2$ and $\|\omega\|_{W^{2,\infty}} \leq 1$. We fix such a test function ω, and we start by rewriting the above expression. For this, let δ denote a positive number to be fixed below (not to be confused with δ_ε), and define $V : (-\delta,\tau+\delta)\times B_\nu(\rho) \to \mathbb{R}^2$ by

$$V(y^0,y^\nu) = \begin{cases} v(0,y^\nu) & \text{if } -\delta < y^0 \leq 0, \\ v(y^0,y^\nu) & \text{if } 0 \leq y^0 \leq \tau, \\ v(\tau,y^\nu) & \text{if } \tau \leq y^0 \leq \tau+\delta. \end{cases}$$

Let $\chi \in C^\infty_c(-\delta,\tau+\delta)$ be a function such that

$$\chi(y^0) \equiv 1 \text{ for } y^0 \in [0,\tau], \qquad \text{and} \qquad \|\chi'\|_\infty \leq C(1+\delta^{-1}).$$

Since $J_\nu V(y^0) = J_\nu v(0)$ for $y^0 \in (-\delta,0]$ and $J_\nu V(y^0) = J_\nu v(\tau)$ for $y^0 \in [\tau,\tau+\delta)$,

$$\begin{aligned}\int_{B_\nu(\rho)} \omega[J_\nu v(\tau,y^\nu) - J_\nu v(0,y^\nu)]\; dy^\nu &= -\int_{-\delta}^{\tau+\delta} \chi'(y^0)(\int_{B_\nu(\rho)} \omega(y^\nu)\, J_\nu V\, dy^\nu)\; dy^0 \\ &= -\int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} (\omega(y^\nu)\,\chi'(y^0)\, dy^0) \wedge \mathbf{J}V.\end{aligned} \tag{5.13}$$

We continue by observing that

$$\omega(y^\nu)\chi'(y^0)\; dy^0 = \omega(y^\nu)d\chi(y^0) = d[\omega(y^\nu)\chi(y^0)] - \chi(y^0)d\omega(y^\nu).$$

Also, since $\mathbf{J}V = d(V^1 \wedge dV^2)$, it is clear that $d\mathbf{J}V = 0$, so that $d(\chi\omega) \wedge \mathbf{J}V = d(\chi\omega \wedge \mathbf{J}V)$, and thus the right-hand side of (5.13) can be rewritten

$$
\begin{aligned}
-\int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} (\omega\, \chi'\, dy^0) \wedge \mathbf{J}V &= \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} \chi\, d\omega \wedge \mathbf{J}V \\
&\qquad - \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} d(\chi\omega) \wedge \mathbf{J}V \\
&= \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} \chi\, d\omega \wedge \mathbf{J}V. \qquad (5.14)
\end{aligned}
$$

2. Properties of ω and the choice of χ imply that

$$|\chi d\omega(y)| \le |y^\nu|^2, \qquad \|D(\chi d\omega)\|_\infty \le C\delta^{-1}.$$

It thus follows from Lemma 10 that

$$
\begin{aligned}
\left| \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} \chi\, d\omega \wedge \mathbf{J}V \right| &\le C|\ln\varepsilon|^{-1} \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} |y^\nu|^2 \left(\frac{1}{2}|DV|^2 + \frac{1}{\varepsilon^2}F(V)\right)\, dy^\nu dy^0 \\
& C\varepsilon^\alpha(1+\delta^{-1}) \left(1 + \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} \left(\frac{1}{2}|DV|^2 + \frac{1}{\varepsilon^2}F(V)\right) dy^\nu dy^0\right).
\end{aligned}
$$

We now fix $\delta := \varepsilon^{\alpha/2}$ and recall the definition of V to find that

$$
\begin{aligned}
\left| \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} \chi\, d\omega \wedge \mathbf{J}V \right| &\le C|\ln\varepsilon|^{-1} \int_{(0,\tau)\times B_\nu(\rho)} (|y^\nu|^2 + \varepsilon^{\alpha/2}) \left(\frac{|v_{y^0}|^2}{2} + e_{\varepsilon,\nu}(v)\right) dy^\nu dy^0 \\
&\quad + C\varepsilon^{\alpha/2}(1 + \int_{\{0\}\times B_\nu(\rho)} e_{\varepsilon,\nu}(v) dy^\nu + \int_{\{\tau\}\times B_\nu(\rho)} e_{\varepsilon,\nu}(v) dy^\nu).
\end{aligned}
$$

The conclusion of the Lemma now follows by recalling (5.13) and (5.14) and renaming α. □

5.2. **proof of Proposition 5.** Now we can give the

Proof of Proposition 5. As in Proposition 3 it suffices to consider smooth solutions v.

To simplify we will write $\mathcal{D}(\cdot)$ and $|||\cdot|||$ instead of $\mathcal{D}(\cdot;\rho_1/2)$ and $|||\cdot|||_{\rho_1/2}$.

Step 1. For simplicity we assume that $s_1 = 0$. We will use the notation $s_{\max} := \min\{\rho_1/2c_*, T_1\}$ and

$$W_\nu(s) := B_\nu(\rho_1 - c_* s), \qquad W(s) := \mathbb{T}^n \times W_\nu(s).$$

We define

$$
\begin{aligned}
\zeta_1(s) &:= \delta_\varepsilon \int_{\{s\}\times W(s)} (1 + \kappa_2 |y^\nu|^2)\, e_\varepsilon(v)\ dy' - 1 \\
\zeta_2(s) &:= \mathcal{D}(v(s)) \\
\zeta_3(s) &:= \delta_\varepsilon \int_{\{s\}\times W(s)} \left[|D_\tau v|^2 + |y^\nu|^2 e_{\varepsilon,\nu}(v)\right]\ dy'.
\end{aligned}
$$

(Recall that κ_2 was fixed in (2.23) and that we took $\kappa_2 = 1$ for $k = 1$.) We first claim that

$$\zeta_1(s) \le \zeta_0 + C \int_0^s \zeta_3(\sigma) d\sigma \qquad \text{for } 0 < s \le s_{max}. \tag{5.15}$$

Indeed, exactly as before we compute that $\zeta_1'(s) = I_1 - c_* I_2$, where

$$I_1 := \delta_\varepsilon \int_{\{s\}\times W(s)} (1 + \kappa_2 |y^\nu|^2) \frac{\partial}{\partial y^0} e_\varepsilon(v) \ dy'$$

$$I_2 = \delta_\varepsilon \int_{\{s\}\times \mathbb{T}^n \times \partial W_\nu(s)} (1 + \kappa_2 |y^\nu|^2) \, e_\varepsilon(v) \ d\mathcal{H}^{N-1}(y').$$

And exactly as before, in I_1 we use the differential inequality (2.27) satisfied by the energy and integrate by parts in the spatial variables. As before, our choice (3.4) of c_* guarantees that the boundary term that arises, involving an integral over $\{s\} \times \mathbb{T}^n \times \partial W_\nu(s)$, is dominated by $-c_* I_2$. This leads as before to the differential inequality

$$\zeta_1' \le C \zeta_3.$$

Since our assumption (5.4) exactly states that $\zeta_1(0) \le \zeta_0$, we conclude that (5.15) holds.

Step 2. Next, we estimate ζ_2. It is clear that $|||\cdot|||$ is a norm, so that $\mathcal{D}_\nu(v(s, y^{\tau\prime})) \le \mathcal{D}_\nu(v(0, y^{\tau\prime})) + |||J_\nu v(s, y^{\tau\prime}) - J_\nu v(0, y^{\tau\prime})|||$ for every $(s, y^{\tau\prime})$, by the triangle inequality. It follows that

$$\begin{aligned} \zeta_2(s) &\le \mathcal{D}(v(0)) + \int_{\mathbb{T}^n} |||J_\nu v(0, y^{\tau\prime}) - J_\nu v(s, y^{\tau\prime})||| dy^{\tau\prime} \\ &\overset{(5.5),(5.11)}{\le} \zeta_0 + C\delta_\varepsilon \int_{\mathbb{T}^n} \int_{(0,s)\times B_\nu(\rho_1/2)} |D_\tau v|^2 + (|y^\nu|^2 + \varepsilon^\alpha) e_{\varepsilon,\nu}(v) dy^\nu dy^0 \ dy^{\tau\prime} \\ &\quad + C\varepsilon^\alpha + C\varepsilon^\alpha \int_{\mathbb{T}^n} \left(\int_{\{0\}\times B_\nu(\rho_1/2)} e_{\varepsilon,\nu}(v) dy^\nu \ + \int_{\{s\}\times B_\nu(\rho_1/2)} e_{\varepsilon,\nu}(v) dy^\nu \right) \ dy^{\tau\prime}. \end{aligned}$$

Also, since $B_\nu(\rho_1/2) \subset W_\nu(s)$ for every $s \le \rho_0/2c_*$, the definitions yield

$$\int_{\mathbb{T}^n} \int_{\{s\}\times B_\nu(\rho_1/2)} e_\varepsilon(v) dy^\nu \ dy^{\tau\prime} \le C\delta_\varepsilon{}^{-1}(\zeta_1(s) + 1) \le C|\ln\varepsilon|(\zeta_1(s) + 1),$$

and similarly for $s = 0$. By combining these and rearranging we find that if $0 \le s \le s_{max}$, then

$$\zeta_2(s) \ \le \zeta_0 + \ C \int_0^s \left[\zeta_3(\sigma) + \varepsilon^\alpha(\zeta_1(\sigma) + C)\right] \ d\sigma + C\varepsilon^\alpha + C\varepsilon^{\alpha/2}(\zeta_0 + \zeta_1(s) + C). \tag{5.16}$$

Step 3. Finally, we show (by *exactly* the same arguments as in the corresponding point of the proof of Proposition 3) that

$$\zeta_3(s) \le C \left(\zeta_1(s) \ + \ \zeta_2(s) \ + \ |\ln\varepsilon|^{-1}\right) \tag{5.17}$$

for every $s \in [0, s_{max}]$. We fix such an s, and we write $v(\cdot)$ instead of $v(s, \cdot)$. It follows from the definitions of ζ_1, ζ_3 and the choice (2.23) of κ_2 that

$$\zeta_1(s) \ge c \ \zeta_3(s) + \ \ \delta_\varepsilon \int_{\{s\}\times W(\rho_1/2c_*)} e_{\varepsilon,\nu}(v) \ dy' \ - \ 1. \tag{5.18}$$

We say that a point $y^{\tau\prime} \in \mathbb{T}^n$ is *good* if $\mathcal{D}_\nu(v(y^{\tau\prime})) \le \kappa_3$ and bad otherwise. Then Chebyshev's inequality and (5.1) imply that $|\{y^{\tau\prime} \in \mathbb{T}^n \ : y^{\tau\prime} \text{ is good } \}| \ge 1 - C\zeta_2(s)$, and exactly as in (3.19), but appealing to Proposition 7 instead of Lemma 5, we infer that

$$\delta_\varepsilon \int_{\{s\}\times W(\rho_1/2c_*)} e_{\varepsilon,\nu}(v)\; dy' \;\ge\; (1 - C\zeta_2(s))\,(1 - C|\ln\varepsilon|^{-1}).$$

Combining this inequality with (5.18), we obtain (5.17).

Step 4. By combining the previous few steps, we see that

$$\zeta_3(s) \le C\zeta_0 + C|\ln\varepsilon|^{-1} + C\int_0^s \zeta_3(\sigma)d\sigma + C\varepsilon^\alpha \int_0^s \int_0^\sigma \zeta_3(t)\; dt\; d\sigma.$$

If we define $\zeta_4(s) := \zeta_3(s) + \zeta_0 + |\ln\varepsilon|^{-1} + \varepsilon^\alpha \int_0^s \zeta_3(\sigma)\; d\sigma$, it follows (since $\zeta_0 \ge \delta_\varepsilon$) that

$$\zeta_4(s) \le C\int_0^s \zeta_4(\sigma)\; d\sigma \quad \forall s \in [0, s_{max}], \qquad \zeta_4(0) \le C\zeta_0.$$

Gronwall's inequality then implies that $\zeta_4(s) \le C\zeta_0$ for all $s \in [0, s_{max}]$. The conclusions of the proposition follow from this together with (5.15) and (5.16). □

5.3. **Proof of Proposition 6.** Finally, we present the proof of Proposition 6. We use notation such as $V^*(s), \partial_i V^*(s)$ and so on, from Section 4.

Proof. As usual, we may assume by an approximation argument, relying on standard well-posedness theory for (2.26), that v is smooth on $\bar V$. Define v^* as in (4.5), and define

$$\begin{aligned}
\zeta_1(s) &= \delta_\varepsilon \int_{\partial_1 V^*(s)} (1 + \kappa_2|y^\nu|^2) e_\varepsilon(v^*) dy' - 1\\
\zeta_2(s) &= \mathcal{D}(v^*(s); \rho_1/2)\\
\zeta_3(s) &= \delta_\varepsilon \int_{\partial_1 V^*(s)} \left[|D_\tau v^*|^2 + |y^\nu|^2 e_{\varepsilon,\nu}(v^*)\right]\; dy'.
\end{aligned}$$

We repeat exactly the arguments of Proposition 5 to find that

$$\zeta_2(s) \;\le\; C\int_{s_0}^s \zeta_3(\sigma) + \varepsilon^\alpha(\zeta_1(\sigma) + C)\; d\sigma + C\varepsilon^\alpha + C\varepsilon^{\alpha/2}(\zeta_0 + \zeta_1(s) + C)$$

and

$$\zeta_3(s) \le C\left(\zeta_1(s) \;+\; \zeta_2(s) \;+\; |\ln\varepsilon|^{-1}\right).$$

To estimate ζ_1, we argue as in the proof of Proposition 4. That is, we apply the divergence theorem to

$$\int_{V(s)} \partial_{y^\alpha}\left[(1 + \kappa_2|y^\nu|^2)\tilde T^\alpha_{\varepsilon,0}\right],$$

where $\tilde T^\alpha_{\varepsilon,\beta}(y) := \delta^\alpha_\beta(\frac{\varepsilon}{2} g^{\gamma\delta} v_{y^\gamma}\cdot v_{y^\delta} + \frac{1}{\varepsilon}F(v)) - \varepsilon\; g^{\alpha\gamma} v_{y^\gamma}\cdot v_{y^\beta}$ and we rewrite, noting that $n_\alpha(y)\tilde T^\alpha_{\varepsilon,0}(y) \ge 0$ for a.e. $y \in \partial_2 V(s)$ exactly as before. This eventually yields

$$\zeta_1(s) \le C\int_{s_0}^s \zeta_3(\sigma)\, d\sigma + A + B,$$

for a.e. $s \in [s_0, s_1]$, where

$$A := \delta_\varepsilon \int_{W_0(s)} (1+\kappa_2|y^\nu|^2)(e_\varepsilon(v) - e_\varepsilon(v_0))(b(y'), y')dy' + \delta_\varepsilon \int_{\partial_0 V(s)} (1+\kappa_2|y^\nu|^2)\, n_i \varphi^i \; d\mathcal{H}^N,$$

$$B := \delta_\varepsilon \int_{[W_1^*(s)\setminus W_1(s)]\cup W_0(s)} (1+\kappa_2|y^\nu|^2) e_\varepsilon(v_0) dy' - 1.$$

We proceed exactly as in the proof of Proposition 4, using Lemmas 3 and 7, the hypotheses (2.34) – (2.36), and elementary arguments to show that $A \le C\zeta_0$ and $B \le C\zeta_0$ for a.e. $s \in [s_0, s_1]$, and hence that $\zeta_1 \le C\int_{s_0}^s \zeta_3(\sigma)d\sigma + C\zeta_0$.

The proof is now finished exactly as in Step 4 of the proof of Proposition 5. □

6. Proof of Theorems 1 and 2

In this section we combine the estimates proved in the previous sections with standard energy estimates in the original (t,x) variables, iterate, and harvest consequences, to complete the proofs of our main results. We mostly give a unified treatment of the cases $k = 1, 2$. To distinguish between the relevant energy densities in the (t,x) and the y variables, in this section we will often use the notation $e_\varepsilon(u;\eta)$ and $e_\varepsilon(v;G)$, see (1.28) and the following discussion.

The following theorem assembles most of our main estimates and will easily imply Theorems 1 and 2; in can be seen as the main result of this paper. In it, and throughout this section, when we write $C(\Gamma, T_0)$, it will denote a constant that may depend upon various choices made in the construction (2.12) of the map ψ that we use to change variables; these choices however are constrained only by Γ and T_0.

Theorem 3. *Let $k = 1$ or 2, $n \ge 1$, and $N = n + k$.*

Let $\Gamma \subset (-T,T)\times\mathbb{R}^N$ be a smooth timelike Minkowski minimal surface of codimension k satisfying our standing assumptions (2.5)- (2.7). Let $u : (-T,T)\times\mathbb{R}^N \to \mathbb{R}^k$ solve (1.1) with initial data satisfying assumpions (2.31), (2.34) - (2.36), for some ζ_0 verifying (2.30).

Given $T_0 < T$, fix $T_1 \in (T_0, T)$ and $\rho_0 > 0$ so small that (2.13), (2.14) and the conclusions of Proposition 1 hold on $(-T_1, T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)$.

Then there exists a constant $C(\Gamma, T_0)$ such that

$$\delta_\varepsilon \int_{[(-T_0,T_0)\times\mathbb{R}^N]\setminus\mathcal{N}} e_\varepsilon(u;\eta)\; dx\, dt \le C\zeta_0, \tag{6.1}$$

for $\mathcal{N} = image(\psi) \cap [(-T_0,T_0)\times\mathbb{R}^N]$; and such that $v = u\circ\psi$ satisfies

$$\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} |D_\tau v|^2 + |y^\nu|^2(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2}F(v))\; dy \le C\zeta_0, \tag{6.2}$$

$$\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \left[(1+\kappa_2|y^\nu|^2) e_\varepsilon(v;G)\right] dy - \mathcal{H}^{1+n}((-T_1,T_1)\times\mathbb{T}^n) \le C\zeta_0, \tag{6.3}$$

(for κ_2 as in (2.23)*, with $\kappa_2 = 1$ when $k = 1$) and*

$$\int_{(-T_1,T_1)\times\mathbb{T}^n} \mathcal{D}_\nu(v(y^\tau);\rho_1/2)\ dy^\tau \ \leq C\zeta_0 \tag{6.4}$$

where $\mathcal{D}_\nu$ was defined in (3.3) *for $k = 1$ and* (5.2) *for $k = 2$, and ρ_1 was found in Lemma 7. Finally,*

$$\|\delta_\varepsilon \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\|_{W^{-1,1}((-T_0,T_0)\times\mathbb{R}^N)} \leq C\sqrt{\zeta_0}. \tag{6.5}$$

The following lemma will be used repeatedly.

Lemma 11. *There exists a constant $C > 0$, depending only Γ, T_1, ρ_0, such that*

$$\frac{1}{C} e_\varepsilon(u;\eta)(\psi(y)) \leq\ e_\varepsilon(v;G)(y)\ \leq\ C e_\varepsilon(u;\eta)(\psi(y))$$

and

$$\frac{1}{C} \leq |\det D\psi(y)| = \sqrt{-g(y)} \leq C$$

for all $y \in (-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)$.

Proof. This is clear from the construction of the diffeomorphism ψ, see in particular (2.20). □

Next, we show that Theorems 1 and 2 follow from directly from Theorem 3 and the above Lemma. The rest of this section will then be devoted to the proof of Theorem 3.

Proofs of Theorems 1 and 2. First we consider the scalar case, ie that of Theorem 1. We define $\mathcal{N}$ as in (2.22). Then as noted in Corollary 1, the function d defined by (2.29) satisfies the eikonal equation (1.12) in $\mathcal{N}$ as required.

Let u solve (1.1) with the initial data given by Lemma 4, in the case $k = 1$, so that it satisfies the assumptions of Theorem 3 with $\zeta_0 = C\varepsilon^2$, and in addition

$$\int_{\mathbb{T}^n\times B_\nu(\rho_0)} (v_0 - q(\frac{y^N}{\varepsilon}))^2 dy' \leq C\varepsilon \tag{6.6}$$

for v_0 as defined in (2.33).

Then conclusion (1.15) of Theorem 1 is exactly (6.5).

To prove (1.14), we recall that $\mathcal{N} \subset \psi((-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0))$, and we use (2.29) and Lemma 11 to estimate

$$\begin{aligned}\delta_\varepsilon \int_{\mathcal{N}} d^2 e_\varepsilon(u;\eta)\,dx\,dt &\leq C\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} (y^\nu)^2 e_\varepsilon(v;G)\,dx\,dt \\ &\overset{(6.3)}{\leq} C\zeta_0 - \left[\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} e_\varepsilon(v;G)dy - \mathcal{H}^{1+n}((-T_1,T_1)\times\mathbb{T}^n)\right]\end{aligned}$$

Next, by using Lemma 5 and arguing exactly as in the the proof of (3.17), we see that

$$2T_1 - \delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} e_\varepsilon(v;G)dy \leq C \int_{(-T_1,T_1)\times\mathbb{T}^n} \mathcal{D}_\nu(v(y^\tau))dy^\tau + Ce^{-c/\varepsilon}. \tag{6.7}$$

The above inequalities and (6.4) imply that $\delta_\varepsilon \int_{\mathcal{N}} d^2 e_\varepsilon(u;\eta)\,dx\,dt \le C\varepsilon^2$. By combining this with (6.1), we obtain (1.14).

Finally, to prove (1.13), note that for every $y' \in \mathbb{T}^n \times B_\nu(\rho_1)$ and every $y^0 \in (-T_1, T_1)$,

$$|v(y^0,y') - v_0(y')| = |v(y^0,y') - v(b(y'),y')| \le |y^0 - b(y')|^{1/2}\left(\int_{-T_1}^{T_1} |\partial_{y^0} v(s,y')|^2 ds\right)^{1/2}.$$

Since $|\partial_{y^0} v| \le |D_\tau v|$, we find by integrating that

$$\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} |v(y^0,y') - v_0(y')|^2\,dy \ \le\ C\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} |D_\tau v|^2 dy \ \stackrel{(6.2)}{\le}\ C\varepsilon$$

using the fact that $\zeta_0 \le C\varepsilon^2$. Then (6.6) implies that

$$\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} |v(y) - q(y^N/\varepsilon)|^2\,dy \le C\varepsilon.$$

By changing variables, using Lemma 11 and recalling (2.29), we obtain (1.13).

The proof of Theorem 2 is essentially the same, except that we do not make any claim about $\int |v(y^0,y') - v_0(y')|^2$, as the estimate $\int |\partial_{y^0} v|^2 dy \le C$ is too weak to provide good control over this quantity.

Otherwise, we follow the above proof. That is, we let u solve (1.1) with the initial data given by Lemma 4, in the case $k=2$, so that it satisfies the assumptions of Theorem 3 with $\zeta_0 = C|\ln\varepsilon|^{-1}$. Then conclusion (1.21) of Theorem 2 is exactly (6.5). To prove (1.20), it suffices in view of (6.1) to prove that

$$\int_{\mathcal{N}} \operatorname{dist}(\cdot,\Gamma)^2 e_\varepsilon(u;\eta)\,dx\,dt \le C.$$

Using Lemma 11 to change variables, and noting that $\operatorname{dist}(\psi(y),\Gamma)^2 \le C|y^\nu|^2$ (since the left-hand side is a smooth function of y that vanishes when $y^\nu = 0$) it suffices to prove that

$$\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} |y^\nu|^2 e_\varepsilon(v;G)\,dy \ \le\ C.$$

This exactly follows the proof of (1.14) in the case $k=1$ above. The estimate corresponding to (6.7) has *exactly* the same form, except that the last term on the right-hand side is now $C|\ln\varepsilon|^{-1}$; this is proved by arguing as before but using Proposition 7 in place of Lemma 5. □

The proof of Theorem 3 will use some standard energy estimates that we now recall.

Lemma 12. *Let $u : \mathbb{R}^{1+N} \to \mathbb{R}^k$ be a smooth, finite-energy solution of the semilinear wave equation* (1.1). *For any $a < b$ and any bounded Lipschitz function $\chi : \mathbb{R}^{1+N} \to \mathbb{R}$*

$$\left|\int_{\{b\}\times\mathbb{R}^N} e_\varepsilon(u;\eta)\chi\ dx - \int_{\{a\}\times\mathbb{R}^N} e_\varepsilon(u;\eta)\chi\ dx\right| \le \int_{(a,b)\times\mathbb{R}^N} e_\varepsilon(u;\eta)|D\chi|\ dx\ dt. \tag{6.8}$$

Also, for any pair a,b of real numbers and any open $A \subset \mathbb{R}^N$

$$\int_{\{b\}\times A_{|b-a|}} e_\varepsilon(u;\eta)\ dx \ \le\ \int_{\{a\}\times A} e_\varepsilon(u;\eta)\ dx, \tag{6.9}$$

where $A_s := \{x \in A : \mathrm{dist}(x, \partial A) > s\}$.

Proof. Both conclusions are standard and follow from the identity $\partial_t e_\varepsilon(u;\eta) = \nabla \cdot (u_t \nabla u)$, satisfied by solutions of (1.1); integration by parts; and the elementary inequality $|u_t \nabla u| \le e_\varepsilon(u;\eta)$. For the second inequality, assuming for concreteness that $a < b$, it is easy to see that the set $\{(t,x) : a < t < b, x \in A_{t-a}\}$ is a set of finite perimeter, so that the divergence theorem holds and there is no problem in justifying the standard argument. □

Lemma 13. *Let* $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}^k$ *be a smooth solution of* (2.26)*. Then for any* $-T_1 \le a < b \le T_1$ *and any* $\chi \in W_0^{1,\infty}((-T_1,T_1) \times \mathbb{T}^n \times B_\nu(\rho_0))$*,*

$$\left| \int_{\{b\}\times\mathbb{R}^N} e_\varepsilon(v;G)\chi \ dy' - \int_{\{a\}\times\mathbb{R}^N} e_\varepsilon(v;G)\chi \ dy' \right| \le C \int_{(a,b)\times\mathbb{R}^N} e_\varepsilon(v;G)\left(|\chi| + |D\chi|\right) \ dy.$$

Proof. Lemma 2 implies that $\partial_{y^0} e_\varepsilon(v;G) \le C e_\varepsilon(v;G) + \nabla \cdot \varphi$, and the positivity (2.16) of the matrix $(a^{\alpha\beta})$, together with the definition (2.28) of φ, implies that $|\varphi| \le C e_\varepsilon(v;G)$. The conclusion follows from these facts together with integration by parts, exactly as in the previous lemma. □

Now we present the

proof of Theorem 3. We treat both cases $k = 1$ and 2 simultaneously. We may assume as usual that u and $v = u \circ \psi$, are smooth.

It is convenient to define $\rho : (-T_0, T_0) \times \mathbb{R}^N \to [0, +\infty]$ by

$$\rho(t,x) = \begin{cases} |y^\nu| & \text{if } (t,x) = \psi(y) \\ +\infty & \text{if } (t,x) \notin \mathcal{N} = \mathrm{image}(\psi). \end{cases}$$

Note that when $k = 1$, $\rho(t,x) = |d(t,x)|$ for $(t,x) \in \mathcal{N}$.

Step 1. Given $u : (-T,T) \times \mathbb{R}^N \to \mathbb{R}^k$, $k = 1$ or 2, solving (1.1), we will say that u is *controlled* on a set $W \subset (-T_0, T_0) \times \mathbb{R}^N$ if there exists a constant C, depending on W, Γ, ψ, such that for any function u satisfying the hypotheses of Theorem 3,

$$\int_W e_\varepsilon(u;\eta) \le C\zeta_0.$$

(We will only say this about sets that are bounded away from Γ). If W is an open set, the integral is understood as $\int \cdots dxdt$, and if W is a subset of some $\{t\} \times \mathbb{R}^N$ then it is understood as $\int \cdots dx$.

Similarly, for a set $W \subset (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$, we say that $v = u \circ \psi$ is controlled on W if there exists a constant $C = C(W, \Gamma, T_0)$ such that

$$\delta_\varepsilon \int_W \left[|D_\tau v|^2 + |y^\nu|^2 (|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2} F(v)) \right] \le C\zeta_0,$$

Again W may be either an open set or a subset of $\{s\} \times \mathbb{T}^n \times B_\nu(\rho_0)$ for some s, and the integral is understood accordingly.

We make some easy remarks. First, if v is controlled on a set W, then since

$$e_\varepsilon(v;G) \le C(\hat\rho)\left[|D_\tau v|^2 + |y^\nu|^2(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2}F(v))\right] \qquad \text{whenever } |y^\nu| \ge \hat\rho,$$

it follows that for any $\hat\rho \in (0,\rho_0)$, $\int_{\{y=(y^\tau,y^\nu)\in W: |y^\nu|\ge\hat\rho\}} e_\varepsilon(v;G) \le C\zeta_0$.

As a result, Lemma 11 implies that if $A \subset \mathrm{Image}(\psi)$ is bounded away from Γ, then u is controlled on A if and only if v is controlled on $\psi^{-1}(A)$,

Finally, we remark that for any $\hat\rho > 0$, the assumptions (2.31), (2.34), (2.35) and a change of variables imply that

$$u \text{ is controlled on } \{(0,x) \in \mathbb{R}^{1+N} : \rho(0,x) \ge \hat\rho\} \tag{6.10}$$

with the implicit constants depending on $\hat\rho$ and ψ, or more precisely on the behavior of ψ on $\{y \in (-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0) : \psi^0(y) = 0\}$.

Step 2. We next claim that for $0 < s_+' < s_+ \le T_1$ and $\rho' < \rho_1/2$,

$$\begin{aligned} &\text{if } v \text{ is controlled on } \{y \in (-T_1,s_+)\times\mathbb{T}^n\times B_\nu(\rho_1/2) : \psi^0(y) > 0\} \\ &\qquad \text{then for every } t\in[0,s_+'],\ u \text{ is controlled on } \{(t,x) \ : \ \rho(t,x) > \rho'\}, \end{aligned} \tag{6.11}$$

and this control is uniform for $t \in [0,s_+']$. To prove this, we fix $\hat\rho > 0$ so small that $\hat\rho \le \rho'$ and

$$\begin{aligned} \{(t,x) : 0<t<s_+', \rho(t,x) < \hat\rho\} &= \{\psi(y) : y \in (-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\hat\rho), \ \ 0 < \psi^0(y) < s_+'\} \\ &\subset \{\psi(y) : y \in (-T_1,s_+)\times\mathbb{T}^n\times B_\nu(\hat\rho) : \psi^0(y) > 0\}. \end{aligned}$$

The point is that if $|y^\nu|$ is small enough and $\psi^0(y) < s_+'$, then $y^0 < s_+$. Such a number $\hat\rho$ exists because $|\psi^0(y^0,y^{\tau\prime},y^\nu) - y^0| \le C|y^\nu|$; this is an easy consequence of the definition of ψ.

Then it follows from the control of v and Step 1 that

$$u \text{ is controlled on } \{(t,x) : \ 0<t<s_+', \ \ \frac{\hat\rho}{2} < \rho(t,x) < \hat\rho\ \}, \tag{6.12}$$

since this set is bounded away from Γ and is contained, by the choice of $\hat\rho$, in the image via ψ of a set on which we have assumed that v is controlled.

Now we fix a function $\chi \in C^\infty([0,s_+']\times\mathbb{R}^N)$ such that $\chi = 1$ wherever $\rho(t,x) > \hat\rho$, and $\chi = 0$ where $\rho(t,x) \le \hat\rho/2$. Then we apply (6.8) with this choice of χ and with $a = 0$ and $b \in (0,s_+')$, and using (6.10) and (6.12), we find that u is controlled on $\{(b,x) \ : \rho(b,x) > \hat\rho\}$, with implicit constants that are uniform for $b \in (0,s_+']$. Thus we have proved (6.11).

Step 3. We next claim that for $0 < s_+ \le T_1$ as above,

$$\begin{aligned} &\text{if } v \text{ is controlled on } \{y \in (-T_1,s_+)\times\mathbb{T}^n\times B_\nu(\rho_1/2) : \psi^0(y) > 0\} \\ &\qquad \text{then } v \text{ is controlled on } \{s_+\}\times\mathbb{T}^n\times[B_\nu(\rho_1)\setminus B_\nu(\frac{\rho_1}{2})]. \end{aligned} \tag{6.13}$$

We first apply (6.11), with parameters $s_+' < s_+$ and $\rho' < \frac12\rho_1$ to be fixed below. We then apply (6.9) with $a = s_+'$ and $|b| \le T$ to conclude that u is controlled on

$$S(s_+',\rho') := \{(t,x) : \ |t| \le T, \ \mathrm{dist}(x, A(s_+',\rho')) > |t - s_+'|\}$$

where
$$A(s'_+, \rho') := \{x \ : \rho(s'_+, x) > \rho'\}.$$
Then Step 1 implies that v is controlled on $\psi^{-1}(S(s'_+, \rho'))$.

Below we will show that we can fix $s'_+ < s_+$ and $\rho' > 0$ such that

$$\psi\left(\{s_+\} \times \mathbb{T}^n \times [B_\nu(\rho_0) \setminus B_\nu(\frac{\rho_1}{4})]\right) \subset\subset S(s'_+, \rho'), \tag{6.14}$$

by which we mean that some open neighborhood of $\psi(\cdots)$ is contained in $S(s'_+, \rho')$. For now we assume that we have selected s'_+ and ρ' so that (6.14) holds, and we complete the proof of (6.13). Indeed, if (6.14) holds, then clearly

$$\{s_+\} \times \mathbb{T}^n \times \left[B_\nu(\rho_0) \setminus B_\nu(\frac{\rho_1}{4})\right] \subset\subset \psi^{-1}(S(s'_+, \rho')).$$

and so there exists some $a < s_+$ such that $(a, s_+)\times\mathbb{T}^n\times\left[B_\nu(\rho_0) \setminus B_\nu(\frac{\rho_1}{4})\right] \subset\subset \psi^{-1}(S(s'_+, \rho'))$. Now we can find some and some smooth nonnegative function χ such that

$$\chi = 1 \text{ on } \{s_+\} \times \mathbb{T}^n \times \left[\bar{B}_\nu(\rho_1) \setminus B_\nu(\frac{\rho_1}{2})\right], \text{ and}$$
$$\mathrm{spt}(\chi) \subset (a, T_1) \times \mathbb{T}^n \times \left[B_\nu(\rho_0) \setminus B_\nu(\frac{\rho_1}{4})\right].$$

Since v is controlled in $\psi^{-1}(S(s'_+, \rho'))$, (6.13) follows from applying Lemma 13 with this choice of χ and a and with $b = s_+$.

Step 4. We next verify (6.14). Since the sets $S(s, \rho)$ depend continuously on s and ρ in an obvious way, (6.14) will follow (for suitable $s'_+ < s_+$ and $\rho' > 0$) if we can show that

$$\psi\left(\{s_+\} \times \mathbb{T}^n \times (B_\nu(\rho_0) \setminus B_\nu(\frac{\rho_1}{4})]\right) \subset\subset S(s_+, 0). \tag{6.15}$$

We will deduce this as a consequence of the following fact: if Σ is a connected spacelike hypersurface in $1 + N$-dimensional Minkowski space, and if we define the solid light cone with vertex (t, x) to be

$$LC(t, x) := \{(t', x') : |x - x'| \geq |t - t'|\}$$

then $LC(t, x) \cap \Sigma = \{(t, x)\}$, for every $(t, x) \in \Sigma$.

To reduce (6.15) to this geometric fact, we define

$$\Sigma := \psi\left(\{s_+\} \times \mathbb{T}^n \times B_\nu(\rho_0)\right).$$

Clearly Σ is a connected hypersurface. We claim that it is also spacelike. To see this, recall (see (2.20)) that $(g_{ij})_{i,j=1}^N$ is positive definite; this implies that $\psi^{-1}(\Sigma) = \{s_+\}\times\mathbb{T}^n\times B_\nu(\rho_0)$ is spacelike with respect to the $(g_{\alpha\beta})$ metric. The claim then follows, since ψ is an isometry between $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$ with the $(g_{\alpha\beta})$ metric and Image$(\psi) \subset \mathbb{R}^{1+N}$ with the Minkowski metric in standard form $ds^2 = -dt^2 + (dx^1)^2 + \ldots + (dx^N)^2$.

Next note that the definition of ψ and the choice (2.13) of ρ_0 imply that

$$\psi\left(\{s_+\} \times \mathbb{T}^n \times [B_\nu(\rho_0) \setminus B_\nu(\frac{\rho_1}{4})]\right) \subset\subset (-T, T) \times \mathbb{R}^N.$$

Thus, in order to prove (6.15) it suffices to show that the closure of $\psi\left(\{s_+\}\times\mathbb{T}^n\times[B_\nu(\rho_0)\setminus B_\nu(\frac{\rho_1}{4})]\right)$ does not intersect $[(-T,T)\times\mathbb{R}^N]\setminus S(s_+,0)$. However, by inspection of the definition of $S(s,\rho)$ one sees that

$$\begin{aligned}[(-T,T)\times\mathbb{R}^N]\setminus S(s_+,0) \;&\subset\; \cup_{\{(x\in\mathbb{R}^N:\rho(s_+,x)=0\}}LC(s_+,x)\\ &=\; \cup_{\{x\in\mathbb{R}^N:(s_+,x)\in\Gamma\}}LC(s_+,x).\end{aligned}$$

In addition, since $\Gamma\cap(\{s_+)\times\mathbb{R}^N)=\psi(\{s_+\}\times\mathbb{T}^n\times\{0\})$, it is clear that

$$\Sigma\;\supset\;\Gamma\cap(\{s_+)\times\mathbb{R}^N).$$

Then the geometric fact mentioned above implies that

$$\begin{aligned}\Sigma\cap\left([(-T,T)\times\mathbb{R}^N]\setminus S(s_+,0)\right)\;&\subset\;\cup_{\{x:(s_+,x)\in\Gamma\}}\Sigma\cap LC(s_+,x)\\ &=\;\Gamma\cap(\{s_+)\times\mathbb{R}^N).\\ &=\;\psi(\{s_+\}\times\mathbb{T}^n\times\{0\}).\end{aligned}$$

Since ψ is injective, this implies that $\psi(\{s_+\}\times\mathbb{T}^n\times[B_\nu(\rho_0)\setminus\{0\}])$ does not intersect $[(-T,T)\times\mathbb{R}^N]\setminus S(s_+,0)$, completing the proof of (6.15).

Step 5. Next we introduce more terminology. For a set $W\subset(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)$, if there exists $W_\tau\subset(-T_1,T_1)\times\mathbb{T}^n$ such that

$$W_\tau\times B_\nu(\rho_1/2)\subset W\subset W_\tau\times B_\nu(\rho_0),$$

then we say that v is *completely controlled* on W if v is controlled on W, and addition there exists a constant $C(W,\psi)$ such that

$$\delta_\varepsilon\int_W\left[(1+\kappa_2|y^\nu|^2)e_\varepsilon(v;G)\right]-\mathcal{H}^{\dim W_\tau}(W_\tau)\le C\zeta_0,$$

and

$$\int_{W_\tau}\mathcal{D}_\nu(v(y^\tau))\;\le C\zeta_0.$$

And as above, we allow W to be either an open set or a subset of some $\{y^0=\text{const}\}$ slice, with the integral understood accordingly, and with $\dim W_\tau=1+n$ in the first case and n in the second.

Now we establish estimates (6.1) - (6.4). First, by assumption $v=u\circ\psi$ satisfies the hypotheses of Proposition 4 (if $k=1$) and Proposition 6 (if $k=2$) and these imply that

(6.16) $\qquad v$ is controlled on $\{y\in(-T_1,s_1)\times\mathbb{T}^n\times B_\nu(\rho_1):\psi^0(y)>0\}$, and

(6.17) $\qquad v$ is completely controlled on $\{s_1\}\times\mathbb{T}^n\times B_\nu(\rho_1)$.

In particular, (6.17) implies that v satisfies the hypotheses of Proposition 3 ($k=1$) or Proposition 5 ($k=2$), with ζ_0 replaced by $C\zeta_0$. These propositions assert that

(6.18) $\qquad v$ is completely controlled on $\{s\}\times\mathbb{T}^n\times B_\nu(\rho_1/2)$, for $s_1\le s\le s_2$.

where $s_2:=\min\{T_1,s_1+\rho_1/2c_*\}$. Then (6.16) and (6.18) imply that v is controlled on $\{y\in(-T_1,s_2)\times\mathbb{T}^n\times B_\nu(\rho_1/2):\psi^0(y)>0\}$. Next we invoke (6.13) to find that v is controlled on $\{s_2\}\times\mathbb{T}^n\times[B_\nu(\rho_1)\setminus B_\nu(\rho_1/2)]$. Hence, appealing again to (6.18), we see that v is completely controlled on $\{s_2\}\times\mathbb{T}^n\times B_\nu(\rho_1)$.

Thus we can apply Proposition 3 or 5 with s_1 replaced by s_2 and ζ_0 multiplied by a suitable constant, but with the same fixed valued of ρ_1 used already. We can repeat this argument as necessary to find, after a finite number of iterations, that v is completely controlled on $\{s\}\times\mathbb{T}^n\times B_\nu(\rho_1/2)$ for $s_1\le s\le T_1$. Since all our energy estimates are clearly valid backwards in the timelike variables, we can also iterate Proposition 3 or 5 backwards, starting from s_1 and arguing as above, to conclude that

$$v \text{ is completely controlled on } \{s\}\times\mathbb{T}^n\times B_\nu(\rho_1/2), \quad -T_1\le s\le T_1. \tag{6.19}$$

Since $T_1>T_0$, we deduce by applying (6.11) in both directions in the t variable that

$$u \text{ is controlled on } \{(t,x)\in(-T_0,T_0)\times\mathbb{R}^N : \rho(t,x)\ge\rho_1/4\}. \tag{6.20}$$

Using these and Lemma 12, we can deduce (arguing as in the proof of (6.11), (6.13), that in fact v is completely controlled on $(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)$. These estimates imply (6.1) — (6.4).

6. It remains to prove (6.5). The point is that it essentially suffices to prove the same estimate in the y variables, in which (6.2) — (6.4) imply a great deal of information about the way in which energy concentrates around Γ, which in these variables is $(-T_1,T_1)\times\mathbb{T}^n\times\{0\}$. We will extract this information using Lemma 5 for the case $k=1$, and an estimate of Kurzke and Spirn [23] for $k=2$.

If $m=(m^\beta_\alpha)$ and $\mathcal{T}=(\mathcal{T}^\alpha_\beta)$, let us write

$$\langle m,\mathcal{T}\rangle := \int m^\beta_\alpha\, d\mathcal{T}^\alpha_\beta.$$

Then we must estimate $\langle m, \delta_\varepsilon\mathcal{T}_\varepsilon(u)-\mathcal{T}(\Gamma)\rangle$ for $(m^\beta_\alpha)\in W^{1,\infty}((-T_0,T_0)\times\mathbb{R}^N)$ with compact support and with $\|m\|_{W^{1,\infty}}\le 1$. To do this, let χ be a smooth function with support in image(ψ), and such that $\chi=1$ on $\{(t,x):|t|<T_0,\rho(t,x)<\rho_0/2\}$. Then

$$\langle m,\mathcal{T}_\varepsilon(u)-\mathcal{T}(\Gamma)\rangle \;=\; \langle(1-\chi)m,\mathcal{T}_\varepsilon(u)\rangle \;+\; \langle\chi m,\mathcal{T}_\varepsilon(u)-\mathcal{T}(\Gamma)\rangle$$

It is clear from the definition (2.8) of $\mathcal{T}_\varepsilon$ that $|\mathcal{T}^\alpha_{\varepsilon,\beta}(u)|\le Ce_\varepsilon(u;\eta)$, so that

$$|\langle(1-\chi)m,\mathcal{T}_\varepsilon(u)\rangle| \le \sum_{\alpha,\beta}\|m^\beta_\alpha\|_\infty\int_{\{(t,x)\in(-T_0,T_0)\times\mathbb{R}^N\ :\rho(t,x)\ge\rho_0/2\}} e_\varepsilon(u;\eta)\,dt\,dx \tag{6.21}$$

$$\le C\zeta_0 \tag{6.22}$$

using (6.1) and (6.2) together with Lemma 11.

7. Let us write $\bar m:=\chi m$. Note that $\bar m$ is supported in image(ψ), and $\|\bar m\|_{W^{1,\infty}}\le C$. We will write

$$\check m^\gamma_\delta(y) \;=\; \bar m^\beta_\alpha\circ\psi(y)\ \psi^\alpha_{y_\delta}(y)\ \phi^\gamma_{x^\beta}\circ\psi(y)\sqrt{-g(y)}, \qquad \phi:=\psi^{-1} \text{ as usual.} \tag{6.23}$$

Note that $\|\check m\|_{W^{1,\infty}}\le C$. We claim that

$$\langle\bar m,\mathcal{T}_\varepsilon(u)\rangle = \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check m^\gamma_\delta(y)\tilde{\mathcal{T}}^\delta_{\varepsilon,\gamma}(v)(y)\ dy \tag{6.24}$$

$$\langle\bar m,\mathcal{T}(\Gamma)\rangle = \int_{(-T_1,T_1)\times\mathbb{T}^n} \check m^\gamma_\delta(y^\tau,0)\ \tilde P^\delta_\gamma\, dy^\tau \tag{6.25}$$

where $\tilde{\mathcal{T}}_\varepsilon(v)$ was defined[8] in (4.11), $\tilde{P}^\delta_\gamma = 1$ if $\delta = \gamma \in \{0,\ldots,n\}$ and 0 otherwise. These are arguably obvious from the tensorial nature of the quantities involved. However, for the convenience of the reader, we note that the definitions (2.8) and (4.11) and (6.23) imply that

$$\bar{m}^\beta_\alpha(t,x)\mathcal{T}^\alpha_{\varepsilon,\beta}(u)(t,x) = \check{m}^\beta_\alpha(y)\tilde{\mathcal{T}}^\alpha_{\varepsilon,\beta}(v)(y)(-g(y))^{-1/2} \qquad \text{for } (t,x) = \psi(y).$$

Then (6.24) follows, from a change of variables, noting that $|\det D\psi| = \sqrt{-g}$, so that $dt\,dx = \sqrt{-g(y)}dy$. To rewrite $\langle \bar{m}, \mathcal{T}(\Gamma)\rangle$, note that our proof of Lemma 1 (to which we refer for notation) showed that

$$\langle \bar{m}, \mathcal{T}(\Gamma)\rangle = \int_{(-T,T)\times\mathbb{T}^n} (\bar{m}^\beta_\alpha \circ H) H^\alpha_{y^a}\eta_{\beta\delta}H^\delta_{y^b}\gamma^{ab}\sqrt{-\gamma}\ dy^\tau$$

where $H : (-T,T)\times\mathbb{T}^n \to (-T,T)\times\mathbb{R}^N$ is the given map parametrizing Γ, see (2.5), and with a,b summed implictly from 0 to n. Since $\psi(y^\tau,0) = H(y^\tau)$, we can rewrite the above integrand in terms of ψ and g, and this leads to (6.25). For this it is useful to note that

$$g_{\alpha\beta}(y^\tau,0) = \begin{cases} \gamma_{\alpha\beta}(y^\tau) & \text{if } \alpha,\beta\le n \\ \delta_{\alpha\beta} & \text{if } \alpha,\beta > n \\ 0 & \text{otherwise.}\end{cases}$$

and that $g^{ab}\ \psi^\delta_{y_b}\ \eta_{\delta\beta} = \phi^a_\alpha\circ\psi\ \eta^{\alpha\gamma}\ \phi^b_\gamma\circ\psi\ \psi^\delta_{y_b}\ \eta_{\delta\beta} = \phi^a_{x^\beta}\circ\psi$ for $a\in\{0,\ldots,n\}$.

8. We now apply our earlier estimates to control various terms in $\langle \bar{m}, \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\rangle$, represented as in (6.24), (6.25) in terms of the y coordinates. In these calculations, we do *not* sum over indices γ and δ when they are repeated.

Case 1: $\gamma\neq\delta$, $\delta\le n$. When this holds,

$$|\tilde{T}^\delta_{\varepsilon,\gamma}| \overset{(4.11)}{=} |g^{\delta\alpha}v_{y^\alpha}v_{y^\gamma}| \overset{(2.19)}{\le} C(|D_\tau v|^2 + |y^\nu|^2|\nabla_\nu v|^2).$$

Thus in this case

$$\delta_\varepsilon\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\gamma_\delta\ \tilde{T}^\delta_{\varepsilon,\gamma}\ dy \overset{(6.2)}{\le} C\|\check{m}\|_\infty\zeta_0.$$

Case 2: $\gamma\neq\delta$, $\gamma\le n$. In this case, we have the weaker estimate

$$|\tilde{T}^\delta_{\varepsilon,\gamma}| \overset{(4.11)}{=} |g^{\delta\alpha}v_{y^\alpha}v_{y^\gamma}| \overset{(2.19)}{\le} C(|D_\tau v|^2 + |D_\tau v||\nabla_\nu v|).$$

So in this case we have

$$\begin{aligned}\delta_\varepsilon\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\gamma_\delta\ \tilde{T}^\delta_{\varepsilon,\gamma}\ dy &\le C\delta_\varepsilon\|\check{m}\|_\infty(\|D_\tau v\|_2^2 + \|D_\tau v\|_2\|\nabla_\nu v\|_2)\\ &\overset{(6.2)}{\le} C\|\check{m}\|_\infty(\zeta_0 + \sqrt{\zeta_0}\sqrt{\delta_\varepsilon}\|\nabla_\nu v\|_2)\\ &\overset{(6.3)}{\le} C\|\check{m}\|_\infty(\zeta_0 + \sqrt{\zeta_0}\sqrt{(\zeta_0+2T_1)}).\end{aligned}$$

Case 3: $\gamma = \delta\le n$. This is the only case in which $\langle m, \mathcal{T}(\Gamma)\rangle$ makes a nonzero contribution.

[8]As remarked earlier $\tilde{\mathcal{T}}_\varepsilon(v)$ is just the energy-momentum tensor for u expressed in terms of the y variables.

In this case, $|g^{\delta\alpha} v_{y^\alpha} v_{y^\delta}| \overset{(2.19)}{\leq} C(|D_\tau v|^2 + |D_\tau v||\nabla_\nu v|)$, so that

$$\begin{aligned}
\tilde{T}^\delta_{\varepsilon,\delta} &= \frac{1}{2} g^{\alpha\beta} v_{y_\alpha} v_{y^\beta} + \frac{1}{\varepsilon^2} F(v) + O(|D_\tau v|^2 + |D_\tau v||\nabla_\nu v|) \\
&\overset{(2.19),(2.20)}{=} \frac{1}{2}|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2} F(v) + O(|D_\tau v|^2 + |D_\tau v||\nabla_\nu v|).
\end{aligned}$$

Thus

$$\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\delta_\delta \, \tilde{\mathcal{T}}^\delta_{\varepsilon,\delta} \, dy = \delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\delta_\delta \; e_{\varepsilon,\nu}(v) \;\, dy + O(\|\check{m}\|_\infty \sqrt{\zeta_0}).$$

The contribution to $\langle \bar{m}, \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\rangle$ from a summand with $\delta = \gamma \leq n$ is thus

$$\begin{aligned}
&\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\delta_\delta \; \delta_\varepsilon e_{\varepsilon,\nu}(v) \;\, dy \\
&\qquad - \int_{(-T_1,T_1)\times\mathbb{T}^n} m^\delta_\delta(y^\tau, 0) \; dy^\tau + O(\|\check{m}\|_\infty \sqrt{\zeta_0}) \\
&= \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \left[\check{m}^\delta_\delta(y^\tau, y^\nu) - \check{m}^\delta_\delta(y^\tau, 0)\right] \; \delta_\varepsilon e_{\varepsilon,\nu}(v) \;\, dy \\
&\qquad - \int_{(-T_1,T_1)\times\mathbb{T}^n} m^\delta_\delta(y^\tau, 0) \left(1 - \delta_\varepsilon \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(v)(y^\tau, y^\nu) dy^\nu\right) \; dy^\tau + O(\|\check{m}\|_\infty \sqrt{\zeta_0}) \\
&=: A + B + O(\|\check{m}\|_\infty \sqrt{\zeta_0}).
\end{aligned}$$

To estimate A, note that $|\check{m}^\delta_\delta(y^\tau, y^\nu) - \check{m}^\delta_\delta(y^\tau, 0)| \leq \|\check{m}\|_{W^{1,\infty}} |y^\nu| \leq C|y^\nu|$, so that

$$|A| \leq \left(\delta_\varepsilon \int |y^\nu|^2 e_{\varepsilon,\nu}(v) \;\, dy)\right)^{1/2} \; \left(\delta_\varepsilon \int e_{\varepsilon,\nu}(v) \;\, dy)\right)^{1/2} \; \leq C\sqrt{\zeta_0}$$

after arguing as in Case 2 above to estimate $\int e_{\varepsilon,\nu}(v) \leq C$.

As for the other term, since $\|\check{m}\|_\infty \leq C$,

$$|B| \leq C \int_{(-T_1,T_1)\times\mathbb{T}^n} |\Theta_1(y^\tau)| \;\, dy^\tau, \qquad \text{for } \Theta_1(y^\tau) := \delta_\varepsilon \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(v)(y^\tau, y^\nu) dy^\nu - 1.$$

Let us say that y^τ is *good* if $\mathcal{D}_\nu(v(y^\tau)) \leq \kappa_3$, where κ_3 is the constant from Lemma 5 and Proposition 7 for $k = 1$ and $k = 2$ respectively. A point will be called *bad* if it is not good. In particular, these results show that if y^τ is good, then

$$\Theta_1(y^\tau) \geq \quad \left\{ \begin{array}{ll} -Ce^{-c/\varepsilon} & \text{if } k = 1 \\ -C|\ln\varepsilon|^{-1} & \text{if } k = 2 \end{array} \right\} \quad \geq -C\zeta_0 \quad \text{ in both cases}$$

since $\delta_\varepsilon \leq \zeta_0$. Thus, since clearly $\Theta_1(y^\tau) \geq -1$ everywhere, we see that

$$|\Theta_1(y^\tau)| \leq \begin{cases} \Theta_1(y^\tau) + C\zeta_0 & \text{if } y^\tau \text{ is good} \\ \Theta_1(y^\tau) + 2 & \text{if } y^\tau \text{ is bad} \end{cases}$$

Thus we compute

$$\begin{aligned}|B| \le\ & C\int_{good\ pts}(\Theta_1(y^\tau)+C\zeta_0)dy^\tau + C\int_{bad\ pts}(\Theta_1(y^\tau)+2)dy^\tau\\ \le\ & C\int_{(-T_0,T_0)\times\mathbb{T}^n}\Theta_1(y^\tau)dy^\tau + C\zeta_0 + 2\mathcal{H}^{1+n}(\{y^\tau\in(-T_0,T_0)\times\mathbb{T}^n\ :\ y^\tau \text{ is bad.}\})\end{aligned}$$

To conclude the estimate, we note that (6.3) implies that $\int_{(-T_0,T_0)\times\mathbb{T}^n}\Theta_1(y^\tau)dy^\tau \le C\zeta_0$, and (6.4) together with Chebyshev's inequality implies that

$$\mathcal{H}^{1+n}(\{y^\tau\in(-T_0,T_0)\times\mathbb{T}^n\ :\ y^\tau \text{ is bad}\}) \le C\int_{(-T_0,T_0)\times\mathbb{T}^n}\mathcal{D}_\nu(v(y^\tau,\cdot))dy^\tau \le C\zeta_0.$$

Thus $|B|\le C\zeta_0$.

Case 4: $\gamma,\delta>n$**.**

Here we consider the cases $k=1,2$ separately.

k=1: Here the assumption of Case 4 reduces to $\gamma=\delta=N$, and we using (2.24) we see that

$$\tilde{\mathcal{T}}^N_{\varepsilon,N} \ =\ \frac12\sum_{a,b=0}^n g^{ab}v_{y^a}v_{y^b} - \frac12(v_{y^N})^2+\frac{1}{\varepsilon^2}F(v) = -\frac12(v_{y^N})^2+\frac{1}{\varepsilon^2}F(v)+O(|D_\tau v|^2).$$

We will write

$$\Theta_2(y^\tau) := \delta_\varepsilon\int_{B_\nu(\rho_0)}\left|v^2_{y^N}-\frac{1}{\varepsilon^2}F(v)\right|dy^\nu,$$

and we will now say that $y^\tau\in(-T_1,T_1)\times\mathbb{T}^n$ is *good* if

$$\mathcal{D}_\nu(v(y^\tau))\le\kappa_3, \qquad \text{and in addition } \Theta_1(y^\tau)\le\kappa_4 \tag{6.26}$$

for κ_3,κ_4 found in Lemma 5. Then Lemma 5 implies that if y^τ is good, then

$$\Theta_2(y^\tau)\le C\sqrt{|\Theta_1(y^\tau)|+\zeta_0}.$$

Thus using Hölder's inequality

$$\begin{aligned}\int_{good\ pts}\Theta_2(y^\tau)dy^\tau &\le \sqrt{C\int_{good\ pts}(|\Theta_1(y^\tau)|+C\zeta_0)dy^\tau}\\ &\le C\sqrt{\zeta_0}\end{aligned}$$

using estimates from Case 3 above. And if y^τ is bad, then

$$\Theta_2(y^\tau)\le C\left(1+\Theta_1(y^\tau)\right)$$

so that

$$\begin{aligned}\int_{bad\ pts}\Theta_2(y^\tau)dy^\tau &\le \sqrt{C\int_{bad\ pts}(\Theta_1(y^\tau)+1)dy^\tau}\\ &\le C\sqrt{\zeta_0}++C\left(\mathcal{H}^{1+n}(\{y^\tau\in(-T_0,T_0)\times\mathbb{T}^n\ :\ y^\tau \text{ is bad}\})\right)^{1/2}\\ &\le C\sqrt{\zeta_0}\end{aligned}$$

where at the end we used Chebyshev's inequality with (6.3), (6.4). Hence

$$|\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}_N^N\ \tilde{\mathcal{T}}_{\varepsilon,N}^N\, dy| \ \le\ C\int_{(-T_0,T_0)\times\mathbb{T}^n} \Theta_2(y^\tau)dy^\tau + O(\zeta_0) \le C\sqrt{\zeta_0}$$

k=2: We claim that when $k=2$,

$$|\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}_\gamma^\delta\ \tilde{\mathcal{T}}_{\varepsilon,\delta}^\gamma\, dy| \le C\sqrt{\zeta_0} \tag{6.27}$$

if $\delta,\gamma\in\{N-1,N\}$. This will complete the proof of (6.5). To prove (6.27), we first note that Proposition 1 implies that

$$\begin{pmatrix} \tilde{\mathcal{T}}_{\varepsilon,N-1}^{N-1} & \tilde{\mathcal{T}}_{\varepsilon,N}^{N-1} \\ \tilde{\mathcal{T}}_{\varepsilon,N-1}^{N} & \tilde{\mathcal{T}}_{\varepsilon,N}^{N} \end{pmatrix} = \begin{pmatrix} \frac12(|v_{y^N}|^2-|v_{y^{N-1}}|^2)+\frac{1}{\varepsilon^2}F(v) & -v_{y^{N-1}}\cdot v_{y^N} \\ -v_{y^{N-1}}\cdot v_{y^N} & \frac12(-|v_{y^N}|^2+|v_{y^{N-1}}|^2)+\frac{1}{\varepsilon^2}F(v) \end{pmatrix}$$
$$+ O(|D_\tau v|^2+|y^\nu|^2|\nabla_\nu v|^2). \tag{6.28}$$

At this point we need Theorem 1 from Kurzke and Spirn [23], which implies that if $w\in H^1(B_\nu(\rho_0),\mathbb{R}^2)$ and

$$\mathcal{D}_\nu(w)\le\kappa_3, \qquad \text{and} \qquad \Theta_1(y^\tau)\le\frac32 \tag{6.29}$$

then

$$\left|\delta_\varepsilon\int_{B_\nu(\rho_0)}\begin{pmatrix} |w_{y^{N-1}}|^2 & w_{y^{N-1}}\cdot w_{y^N} \\ w_{y^{N-1}}\cdot v_{y^N} & |w_{y^N}|^2\end{pmatrix}dy^\nu - \begin{pmatrix}1&0\\0&1\end{pmatrix}\right|$$
$$\le C\left(\delta_\varepsilon\int_{B_\nu(\rho_0)} e_{\varepsilon,\nu}(w)dy^\nu - 1 + C\delta_\varepsilon\right)^{1/2}.$$

(The main hypothesis of the Kurzke-Spirn estimate is (5.9), and we have shown in the proof of Proposition 7 that this follows from (6.29).) Accordingly, we will continue to say (exactly parallel to the case $k=1$, see (6.26)) say that $y^\tau\in(-T_1,T_1)\times\mathbb{T}^n$ is *good* if $v(y^\tau,\cdot)\in H^1(B_\nu(\rho_0);\mathbb{R}^2)$ satisfies (6.29). As usual, a point that is not good is said to be *bad*. It follows as usual from Chebyshev's inequality and (6.3), (6.4) that

$$\mathcal{H}^{1+n}(\{y^\tau\in(-T_0,T_0)\times\mathbb{T}^n\ :\ y^\tau \text{ is bad}\})\le C\zeta_0.$$

The Kurzke-Spirn inequality implies that if y^τ is good, then

$$\frac{\delta_\varepsilon}{2}\int_{B_\nu(\rho)}|\nabla_\nu v(y^\tau,y^\nu)|^2dy^\nu \ge 1 - C\left(\Theta_1(y^\tau)+C\delta_\varepsilon\right)^{1/2}.$$

Thus for a good point y^τ,

$$\delta_\varepsilon\int_{B_\nu(\rho_0)}\frac{1}{\varepsilon^2}F(v)(y^\tau,y^\nu)dy^\nu = \Theta_1(y^\tau)+\left(1-\frac{\delta_\varepsilon}{2}\int_{B_\nu(\rho)}|\nabla_\nu v(y^\tau,y^\nu)|^2dy^\nu\right)$$
$$\le\Theta_1(y^\tau)+C\left(\Theta_1(y^\tau)+C\delta_\varepsilon\right)^{1/2}.$$

Similarly, the Kurzke-Spirn estimate also implies that if y^τ is good, then

$$\delta_\varepsilon\left|\int_{B_\nu(\rho_0)}(|v_{y^N}|^2-|v_{y^{N-1}}|^2)\ dy^\nu\right|+\delta_\varepsilon\left|\int_{B_\nu(\rho_0)}v_{y^N}\cdot v_{y^{N-1}}\ dy^\nu\right|\le C(\Theta_1(y^\tau)+C\delta_\varepsilon)^{1/2}$$

Combining these and recalling (6.28), we see that if y^τ is good, then for $\gamma,\delta\in\{N-1,N\}$,

$$\left|\delta_\varepsilon\int_{B_\nu(\rho_0)}\mathcal{T}^\delta_{\varepsilon,\gamma}(y^\tau,y^\nu)dy^\nu\right|\le C(\Theta_1(y^\tau)+C\delta_\varepsilon)^{1/2}+\Theta_3(y^\tau) \tag{6.30}$$

where $\Theta_3(y^\tau):=\delta_\varepsilon\int_{B_\nu(\rho_0)}(|D_\tau v|^2+|y^\nu|^2|\nabla_\nu v|^2)dy^\nu$. Also, if y^τ is bad, then (6.28) implies that

$$\delta_\varepsilon\int_{B_\nu(\rho_0)}\left|\mathcal{T}^\delta_{\varepsilon,\gamma}(y^\tau,y^\nu)\right|dy^\nu\le C(\Theta_1(y^\tau)+1).$$

This last fact together with the estimate of the size of the bad set and (6.3) implies that

$$\left|\delta_\varepsilon\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)}\check{m}^\delta_\gamma\ \tilde{\mathcal{T}}^\gamma_{\varepsilon,\delta}\,dy\right|\le|A|+|B|+C\zeta_0,$$

where

$$A:=\delta_\varepsilon\int_{good\ \ pts\in(-T_1,T_1)\times\mathbb{T}^n}\int_{B_\nu(\rho)}\check{m}^\delta_\gamma(y^\tau,0)\ \tilde{\mathcal{T}}^\gamma_{\varepsilon,\delta}\,dy^\nu dy^\tau$$
$$B:=\delta_\varepsilon\int_{good\ \ pts\in(-T_1,T_1)\times\mathbb{T}^n}\int_{B_\nu(\rho)}[\check{m}^\delta_\gamma(y^\tau,y^\nu)-\check{m}^\delta_\gamma(y^\tau,0)]\ \tilde{\mathcal{T}}^\gamma_{\varepsilon,\delta}\,dy^\nu dy^\tau$$

From (6.30), Hölder's inequality, (6.2), and (6.3), we see that

$$|A|\le\|\check{m}\|_\infty\int_{good\ \ pts\in(-T_1,T_1)\times\mathbb{T}^n}C[(\Theta_1(y^\tau)+C\delta_\varepsilon)^{1/2}+\Theta_3(y^\tau)]dy^\tau\ \le\ C\sqrt{\zeta_0}.$$

And since $\|\hat{m}\|_{W^{1,\infty}}\le C$,

$$|B|\le C\delta_\varepsilon\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)}|y^\nu|e_{\varepsilon,\nu}(v)\ dy.$$

We have shown in Case 3 above that the right-hand side above is bounded by $C\sqrt{\zeta_0}$, so we find that $|A|+|B|\le C\sqrt{\zeta_0}$. We therefore have proved (6.27), and (6.5) follows. □

7. Appendix

In this appendix we give the proof of Proposition 1. Recall that we have defined $G=D\psi^T\,\eta\,D\psi$, where $\psi(y^\tau,y^\nu):=\ H(y^\tau)+\sum_{i=1}^k\bar\nu_i(y^\tau)\,y^{n+i}$. Here H is the given parametrization of the minimal surface Γ, and the vectors $\{\bar\nu_i\}$ form an orthonormal frame for the normal bundle of Γ, see (2.11). The proposition asserts certain properties of $g:=\det G$ and $(g^{ij})=G^{-1}$. We will use the following lemma to read off properties of G^{-1} from those of G.

Lemma 14. *Let M be a matrix written in block form as*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(where the blocks need not be of equal size, ie $A \in M^{n\times n}, B \in M^{n\times m}, C \in M^{m\times n}$ and $D \in M^{m\times m}$ for some m, n. Assume that

$$N = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \tag{7.1}$$

is well-defined. Then $N = M^{-1}$.

This is proved by simply verifying that that $MN = I$.

Proof of Propositions 1 and 2. We will think of $\nu(y^\tau)$ as a $(1+N)\times k$ matrix with columns $\bar\nu_i$, $i = 1, ..., k$, and of y^ν as a $k \times 1$ vector, so that $\nu y^\nu := \sum_{i=1}^k \bar\nu_i(y^\tau)\, y^{n+i}$.

Step 1. To start, note that $\nabla_\nu \psi(y) = \nu(y^\tau)$, so the choice (2.11) of ν implies that G can be written in block form as

$$G = \begin{pmatrix} G_{\tau\tau} & G_{\tau\nu} \\ G_{\nu\tau} & I_k \end{pmatrix}$$

where

$$G_{\tau\tau} := D_\tau\psi^T\,\eta\,D_\tau\psi \in M^{(1+n)\times(1+n)} \qquad \text{and } G_{\tau\nu} = G_{\nu\tau}^T := D_\tau(\nu y^\nu)^T\,\eta\nu \in M^{(1+n)\times k}$$

and I_k denotes the $k \times k$ identity matrix. Observe that

$$|G_{\tau\nu}| \le C|y^\nu| \quad \text{when } k \ge 2 \qquad \text{and } G_{\tau\nu} \equiv 0 \quad \text{for } k = 1. \tag{7.2}$$

The second assertion above follows from differentiating the identity $(\nu^T\eta\nu)(y^\tau) \equiv 1$. Since $\psi(y^\tau, 0) = H(y^\tau)$, we can write $G_{\tau\tau}(y^\tau, 0)$ in block form as

$$G_{\tau\tau}(y^\tau, 0) = \begin{pmatrix} H_{y_0}^T\eta H_{y_0} & H_{y_0}^T\eta\nabla H \\ \nabla H^T\eta H_{y_0} & \nabla H^T\eta\nabla H \end{pmatrix} = \begin{pmatrix} -1 + |h_{y^0}|^2 & 0 \\ 0 & \nabla h^T\nabla h \end{pmatrix}$$

where we have used (2.5), (2.6). It then follows from (2.7) and the smoothness of H that $G_{\tau\tau}(y^\tau, 0)$ is invertible, with uniformly bounded inverse, for $y^\tau \in [-T_1, T_1] \times \mathbb{T}^n$. It follows by continuity that that $G_{\tau\tau}(y)$ is invertible, with uniformly bounded inverse, for $y \in [-T_1, T_1] \times \mathbb{T}^n \times B_\nu(\rho_0)$, if ρ_0 is chosen small enough.

Step 2. Next, we note that $G_{\tau\tau}(y) = G_{\tau\tau}(y^\tau, 0) + O(|y^\tau|)$, and we use (7.1) and (7.2) to find that, taking ρ_0 smaller if necessary, $G(y)$ is invertible for $y \in [-T_1, T_1]\times\mathbb{T}^n\times B_\nu(\rho_0)$, with

$$\begin{aligned} G^{-1}(y) &= \begin{pmatrix} (G_{\tau\tau}(y^\tau,0) + O(|y^\nu|))^{-1} & O(|y^\nu|) \\ O(|y^\nu|) & (I_k - O(|y^\nu|^2))^{-1} \end{pmatrix} \\ &= \begin{pmatrix} G_{\tau\tau}^{-1}(y^\tau, 0) & 0 \\ 0 & I_k \end{pmatrix} + \begin{pmatrix} O(|y^\nu|) & O(|y^\nu|) \\ O(|y^\nu|) & O(|y^\nu|^2) \end{pmatrix}. \end{aligned} \tag{7.3}$$

We have used (more than once) the fact that $G_{\tau\tau}^{-1}(y^\tau, 0)$ is uniformly bounded, which implies that $|(G_{\tau\tau} + A)^{-1} - G_{\tau\tau}^{-1}| \le C|A|$ for A sufficiently small, with a uniform constant C.

From (7.3) and (2.7), we easily conclude that (2.20), (2.19), and the first estimate of (2.17) hold. Moreover, if $k = 1$ then, in view of (7.2),

$$G^{-1}(y) \;=\; \begin{pmatrix} G_{\tau\tau}(y)^{-1} & 0 \\ 0 & I_k \end{pmatrix} = G^{-1}(y) \;=\; \begin{pmatrix} G_{\tau\tau}(y^\tau,0)^{-1} + O(|y^\nu|) & 0 \\ 0 & I_k \end{pmatrix}$$

from which we infer (2.24) and (2.25).

To establish the second conclusion of (2.17), we differentiate the identity $G^{-1}G = I$ to find that

$$G^{-1}_{y_0} = -G^{-1}\, G_{y_0}\, G^{-1}.$$

Our earlier expression for G implies that

$$G_{y_0} = \begin{pmatrix} G_{\tau\tau,y_0} & O(|y^\nu|) \\ O(|y^\nu|) & 0 \end{pmatrix}$$

and one can readily check that this implies that $|g^{\alpha\beta}_{y^0}\xi_\alpha\xi_\beta| \le C(|\xi_\tau|^2 + |y^\nu|^2\,|\xi_\nu|^2)$, which completes the proof of (2.17).

Step 3 It remains to establish (2.18). To do this, fix $\zeta \in C^\infty_0([-T_1,T_1]\times\mathbb{T}^n;\mathbb{R}^k)$, and for $\sigma\in\mathbb{R}$ define

$$f(\sigma) = \int_V \big(-\det(D_\tau H_\sigma^T\ \eta\ DH_\sigma)\big)^{1/2}\, dy^\tau, \tag{7.4}$$

where

$$H_\sigma(y^\tau) = H(y^\tau) + \sigma\nu(y^\tau)\zeta(y^\tau) \;=\; \psi(y^\tau,\sigma\zeta(y^\tau)).$$

Note that for σ small, H_σ parametrizes a surface Γ_σ that is a small variation of the original surface Γ. Because Γ is a Minkowski minimal surface, it follows that $f'(0) = 0$. We will show that this yields the conclusion of the lemma.

Thinking of $D\zeta$ as a $k\times(1+n)$ matrix, a direct computation yields

$$DH_\sigma(y^\tau) = D_\tau\psi(y^\tau,\sigma\zeta(y^\tau)) + \sigma\nu(y^\tau)D\zeta(y^\tau)$$

It then follows from (2.11) that $DH_\sigma^T\ \eta\ DH_\sigma$ has the form

$$[DH_\sigma^T\,\eta\,DH_\sigma](y^\tau) \;=\; [D_\tau\psi^T\,\eta\,D_\tau\psi](y^\tau,\sigma\zeta(y^\tau)) + \sigma^2 B(y^\tau).$$

for some matrix $B(y^\tau)$ that depends smoothly on y^τ. Since

$$\frac{d}{d\sigma}\det(A(\sigma)+\sigma^2 B)\Big|_{\sigma=0} \;=\; \frac{d}{d\sigma}\det A(\sigma)\Big|_{\sigma=0}$$

if $A(\sigma)$ are square matrices depending smoothly on a real parameter σ, it follows that

$$\frac{d}{d\sigma}\det(DH_\sigma^T\ \eta\ DH_\sigma)(y^\tau) = \frac{d}{d\sigma}\det(D_\tau\psi^T\ \eta\ D_\tau\psi)(y^\tau,\sigma\zeta(y^\tau)) \tag{7.5}$$

at $\sigma = 0$.

Step 4. We next note that

$$\det(D_\tau\psi^T\ \eta\ D_\tau\psi)(y^\tau,\sigma\zeta(y^\tau)) = \det(D\psi^T\ \eta\ D\psi)(y^\tau,\sigma\zeta(y^\tau)) + O(\sigma^2). \tag{7.6}$$

Indeed, this follows by rather easy linear algebra considerations from the fact that

$$D\psi^T\ \eta\ D\psi(y^\tau,\sigma\zeta(y^\tau)) = \begin{pmatrix} D_\tau\psi^T\ \eta\ D_\tau\psi & O(\sigma) \\ O(\sigma) & I_k + O(\sigma^2) \end{pmatrix}. \tag{7.7}$$

By combining (7.5) and (7.6) that

$$\frac{d}{d\sigma}\det(DH_\sigma^T\ \eta\ DH_\sigma)(y^\tau)\Big|_{\sigma=0} = \frac{d}{d\sigma}g(y^\tau,\sigma\zeta(y^\tau))\Big|_{\sigma=0} = \nabla_\nu g(y^\tau,0)\cdot\zeta$$

at $\sigma = 0$. Also, it follows from (2.7) and continuity that $\det(DH_\sigma^T\ \eta\ DH_\sigma)(y^\tau)$ and $g(y^\tau,\sigma\zeta(y^\tau))$ are bounded away from 0 for ζ small enough, and hence

$$\frac{d}{d\sigma}\left(-\det(DH_\sigma^T\ \eta\ DH_\sigma)(y^\tau)\right)^{1/2}\Big|_{\sigma=0} = \nabla_\nu\sqrt{-g}\cdot\zeta$$

Thus the identity $f'(0) = 0$ reduces to

$$0 = \int \nabla_\nu\sqrt{-g}\cdot\zeta\ \ dy^\tau$$

Since ζ is arbitrary, we conclude that $\nabla_\nu\sqrt{-g} = 0$. This fact and (7.3) imply the required estimate (2.18). □

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Department of Mathematics, University of Toronto, Toronto, Canada M5S 2E4
E-mail address: rjerrard@math.toronto.edu